Finding a Path in Group-Labeled Graphs with Two Labels Forbidden*

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Abstract

The parity of the length of paths and cycles is a classical and well-studied topic in graph theory and theoretical computer science. The parity constraints can be extended to the label constraints in a group-labeled graph, which is a directed graph with a group label on each arc. Recently, paths and cycles in group-labeled graphs have been investigated, such as finding non-zero disjoint paths and cycles.

In this paper, we present a solution to finding an s–t path in a group-labeled graph with two labels forbidden. This also leads to an elementary solution to finding a zero path in a $\mathbb{Z}_3$-labeled graph, which is the first nontrivial case of finding a zero path. This situation in fact generalizes the 2-disjoint paths problem in undirected graphs, which also motivates us to consider that setting. More precisely, we provide a polynomial-time algorithm for testing whether there are at most two possible labels of s–t paths in a group-labeled graph or not, and finding s–t paths attaining at least three distinct labels if exist. We also give a necessary and sufficient condition for a group-labeled graph to have exactly two possible labels of s–t paths, and our algorithm is based on this characterization.

Keywords Group-labeled graph, Non-zero path, s–t path, 2-disjoint paths.

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1 Introduction

1.1 Background

The parity of the length of paths and cycles in a graph is a classical and well-studied topic in graph theory and theoretical computer science. As the simplest example, one can easily check the bipartiteness of a given undirected graph, i.e., we can determine whether it contains a cycle of odd length or not. This can be done in polynomial time also in the directed case by using the ear decomposition. It is also an important problem to test whether a given directed graph contains a directed cycle of even length or not, which is known to be equivalent to Pólya's permanent problem [12] (see, e.g., [11]). A polynomial-time algorithm for this problem was devised by Robertson, Seymour, and Thomas [14].

In this paper, we focus on paths connecting two specified vertices s and t. It is easy to test whether a given undirected graph contains an s–t path of odd (or even) length or not, whereas the same problem is NP-complete in the directed case [10] (follows from [5]). A natural generalization of this problem is to consider paths of length p modulo q. One can easily see that, when $q = 2$, both of the following problems generalize the problem of finding an odd (or even) s–t path in an undirected graph:

- finding an s–t path of length p modulo q in an undirected graph, and
- finding an s–t path whose length is NOT p modulo q in an undirected graph, which is equivalent to determining whether all s–t paths are of length p modulo q or not.

Although these two generalizations are similar to each other, they are essentially different in the case of $q \geq 3$. In fact, a linear-time algorithm for the second generalization was given by Arkin, Papadimitriou, and Yannakakis [1] for any q, whereas not so much was known about the first generalization.

Recently, as another generalization of the parity constraints, paths and cycles in a group-labeled graph have been investigated, where a group-labeled graph is a directed graph with each arc labeled by a group element. In a group-labeled graph, the label of a walk is defined as the sum (or the ordered product when the underlying group is non-abelian) of the labels of the traversed arcs, where each arc can be traversed in the converse direction and then the label is inversed (see Section 2.1 for the precise definition). Analogously to paths of length p modulo q, it is natural to consider the following two problems: for a given element α,

(I) finding an s–t path of label α in a group-labeled graph, and

(II) finding an s–t path whose label is NOT α in a group-labeled graph, which is equivalent to determining whether all s–t paths are of label α or not.

Note that, when we consider Problem (I) or (II), by changing uniformly the labels of the arcs incident to s if necessary, we may assume that α is the identity of the underlying group. Hence, each problem is equivalent to finding a zero path or a non-zero path in a group-labeled graph. In what follows, we assume the black-box access to the underlying group, i.e., we can perform elementary operations for it in constant time (see Section 2.1 for the precise assumption).

If the underlying group is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = (\{0, 1\}, +)$ and the label of each arc is 1, then the label of a path corresponds to the parity of its length because $-1 = 1$ in $\mathbb{Z}_2$. This shows that both of these two problems generalize the problem of finding an odd (or even) s–t path in an undirected graph. We note that, in a $\mathbb{Z}_2$-labeled graph, finding an s–t path of label $\alpha \in \mathbb{Z}_2$ is equivalent to finding an s–t path whose label is not $\alpha + 1 \in \mathbb{Z}_2$, but such equivalence cannot hold for any other nontrivial group.

As shown in Section 2.2, Problem (II) can be reduced to testing whether a group-labeled graph contains a non-zero cycle, whose label is not the identity. With this observation, Problem (II) can be easily solved in polynomial time for any underlying group. We mention that there are several results for packing non-zero paths [2,3,18,20] and non-zero cycles [9,19] with some conditions.

On the other hand, the difficulty of Problem (I) is heavily dependent on the underlying group Γ. When $\Gamma \simeq \mathbb{Z}_2$, since Problems (I) and (II) are equivalent as discussed above, it can be easily solved in polynomial time. When $\Gamma = \mathbb{Z}$, Problem (I) is NP-complete since the directed s–t Hamiltonian path problem reduces to this problem by labeling each arc with $1 \in \mathbb{Z}$ and letting $\alpha := n-1 \in \mathbb{Z}$, where n denotes the number of vertices. Huynh [8] showed the polynomial-time solvability of Problem (I) for any fixed finite abelian group, which is deeply dependent on the graph minor theory.

To investigate the gap between Problems (I) and (II), we make a new approach to these problems by generalizing Problem (II) so that multiple labels are forbidden. In this paper, we provide a solution to the case when two labels are forbidden. Our result also leads to an elementary solution to the first nontrivial case of Problem (I), i.e., when $\Gamma \simeq \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} = (\{0, \pm 1\}, +)$.

1.2 Our contribution

Let Γ be an arbitrary group. For a Γ-labeled graph G and two distinct vertices s and t, let $l(G; s, t)$ be the set of all possible labels of s–t paths in G. Our first contribution is to give a characterization of Γ-labeled graphs G with two specified vertices s, t such that $l(G; s, t) = \{\alpha, \beta\}$, where α and β are distinct elements in Γ. Roughly speaking, we show that $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is obtained from "nice" planar graphs (and some trivial graphs) by "gluing" them together (see Section 3.3). It is interesting that the planarity, which is a topological condition, appears in the characterization.

There exists an easy characterization of triplets (G, s, t) with $|l(G; s, t)| = 1$, which is used to solve Problem (II) (see Section 2.2 for details). Our characterization leads to the first nontrivial classification of Γ-labeled graphs in terms of the possible labels of s–t paths, and the classification is complete when $\Gamma \simeq \mathbb{Z}_3$.

We also show an algorithmic result, which is our second contribution. Based on the fact that our characterization can be tested in polynomial time, we present a polynomial-time algorithm for testing whether $|l(G; s, t)| \leq 2$ or not and finding at least three s–t paths whose labels are distinct if exist (see Theorem 9). In particular, our algorithm leads to an elementary solution to Problem (I) when $\Gamma \simeq \mathbb{Z}_3$, i.e., for each $\alpha \in \mathbb{Z}_3$, we can test whether $\alpha \in l(G; s, t)$ or not, and find an s–t path of label α if exists.

Note again that our results are not dependent on Γ, which can be non-abelian or infinite (as long as we can efficiently perform elementary operations for Γ).

1.3 k-disjoint paths problem

Problem (I) in a $\mathbb{Z}_3$-labeled graph in fact generalizes the 2-disjoint paths problem, which also motivates us to consider the situation when two labels are forbidden. The 2-disjoint paths problem is to determine whether there exist two vertex-disjoint paths such that one is from s_1 to t_1 and the other from s_2 to t_2 for distinct vertices s_1, s_2, t_1, t_2 in a given undirected graph. We can reduce the 2-disjoint paths problem to Problem (I) in a $\mathbb{Z}_3$-labeled graph as follows: let $s := s_1$ and $t := t_2$, replace every edge in the given graph with an arc with label 0, add one arc from t_1 to s_2 with label 1, and ask whether the constructed $\mathbb{Z}_3$-labeled graph contains an s–t path of label 1 or not. If the answer is YES, then there exist desired two disjoint paths, and otherwise there do not.

The 2-disjoint paths problem can be solved in polynomial time [15–17], and the following theorem characterizes the existence of two disjoint paths.

Theorem 1 (Seymour [16]). *Let $G = (V, E)$ be an undirected graph and $s_1, t_1, s_2, t_2 \in V$ distinct vertices. Then, there exist two vertex-disjoint paths P_i connecting s_i and t_i ($i = 1, 2$) if and only if there is no family of disjoint vertex sets $X_1, X_2, \ldots, X_k \subseteq V \setminus \{s_1, t_1, s_2, t_2\}$ such that*

1. *$N_G(X_i) \cap X_j = \emptyset$ for distinct $i, j \in \{1, 2, \ldots, k\}$,*
2. *$|N_G(X_i)| \leq 3$ for $i = 1, 2, \ldots, k$, and*
3. *if G' is the graph obtained from G by deleting X_i and adding a new edge joining each pair of distinct vertices in $N_G(X_i)$ for each $i \in \{1, 2, \ldots, k\}$, then G' can be embedded on a plane so that s_1, s_2, t_1, t_2 are on the outer boundary in this order.*

Our characterization (Theorem 12) of group-labeled graphs with exactly two possible labels of s–t paths is inspired by Theorem 1, and we use this theorem in the proof.

We next mention that the k-disjoint paths problem can also be regarded as a special case of Problem (I) for any fixed integer $k \geq 2$. This observation was in fact described in [8, p. 11]. However, their reduction is inadequate, which cannot distinguish two pairs of disjoint s_i–t_i path and s_j–t_j path and disjoint s_i–s_j path and t_i–t_j path for any distinct i, j.

The k-disjoint paths problem is, for a given undirected graph with $2k$ distinct vertices s_i, t_i ($i = 1, 2, \ldots, k$), to determine whether there exist k vertex-disjoint paths such that each path connects s_i and t_i. This problem can be formulated as Problem (I) using the alternating group A_{2k-1} (which is indeed isomorphic to $\mathbb{Z}_3$ when $k = 2$) as follows: replace each edge with an arc with label $\mathrm{id} \in A_{2k-1}$, add an arc from t_i to s_{i+1} with label $(2i-1\ \ 2i+1\ \ 2i) \in A_{2k-1}$ for each $i = 1, 2, \ldots, k-1$, and ask whether there exists an s_1–t_k path of label

$$\sigma := (2k-3\ \ 2k-1\ \ 2k-2) \cdots (3\ 5\ 4)(1\ 3\ 2)$$

or not. It is easy to check that σ is the unique permutation mapping 1 to $2k-1$ which can be constructed in such an A_{2k-1}-labeled graph.

Although the k-disjoint paths problem can be solved in polynomial time for fixed k [13], its solution requires sophisticated arguments based on the graph minor theory. This suggests that Problem (I) is a challenging problem even if the size of the underlying group is bounded.

1.4 Organization

The rest of this paper is organized as follows. In Section 2, we define several terms, notations, and operations, and describe well-known results. Section 3 is devoted to presenting our results: the efficient solvability of the problem to find an s–t path with two labels forbidden, and a characterization of Γ-labeled graphs with exactly two possible labels of s–t paths. Their verifications are shown in Sections 4 and 5, which provide a concrete description of a polynomial-time algorithm for the problem with its correctness and a proof of our characterization, respectively.

2 Preliminaries

2.1 Terms and notations

Throughout this paper, let Γ be a group (which can be non-abelian or infinite), for which we usually use multiplicative notation with denoting the identity by 1_Γ (we sometimes use additive notation with denoting the identity by 0, e.g., when $\Gamma \simeq \mathbb{Z}_3$). We assume that elementary operations for Γ can be performed, i.e., the following procedures can be done in constant time for any $\alpha, \beta \in \Gamma$: getting the inverse element $\alpha^{-1} \in \Gamma$, computing the product $\alpha\beta \in \Gamma$, and testing the identification $\alpha = \beta$. A directed graph $G = (V, E)$ with a mapping $\psi_G \colon E \to \Gamma$ (called a *label function*) is called a Γ*-labeled graph.*

2.1.1 Graphs

Let $G = (V, E)$ be a directed graph. For vertices $v_0, v_1, \ldots, v_l \in V$ and arcs $e_1, e_2, \ldots, e_l \in E$ with $e_i = v_{i-1}v_i$ or $e_i = v_i v_{i-1}$ $(i = 1, 2, \ldots, l)$, a sequence $W = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ is called a *walk* in G. A walk W is called a *path* (in particular, a *v_0–v_l path*) if $v_0, v_1, \ldots, v_l$ are distinct, and a *cycle* if $v_0, v_1, \ldots, v_{l-1}$ are distinct and $v_0 = v_l$. We call v_0 and v_l (which may coincide) the *end vertices of* W, and each v_i $(1 \leq i \leq l-1)$ an *inner vertex on* W. For i, j with $0 \leq i < j \leq l$, let $W[v_i, v_j]$ denote the subwalk $(v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j)$ of W. Let $\bar{W}$ denote the reversed walk of W, i.e., $\bar{W} = (v_l, e_l, \ldots, v_1, e_1, v_0)$.

Let $X \subseteq V$ be a vertex set. We denote by $\delta_G(X)$ the set of arcs between X and $V \setminus X$ in G and by $N_G(X)$ the set of vertices adjacent to X in G, i.e., $\delta_G(X) := \{ e = xy \in E \mid |\{x, y\} \cap X| = 1 \}$ and $N_G(X) := \{ y \in V \setminus X \mid \delta_G(X) \cap \delta_G(\{y\}) \neq \emptyset \}$. We denote a singleton $\{x\}$ by its element x when it makes no confusion.

Let $G[X] := (X, E(X))$ denote the subgraph of G *induced by* X, where $E(X) := \{ e = xy \in E \mid \{x, y\} \subseteq X \}$. We denote by $G - X$ the subgraph of G obtained by removing all vertices in X, i.e., $G - X = G[V \setminus X]$. For an arc set $F \subseteq E$, we also denote by $G - F$ the subgraph of G obtained by removing all arcs in F, i.e., $G - F = (V, E \setminus F)$. Define $G[\![X]\!] := G[X \cup N_G(X)] - E(N_G(X))$.

For an integer $k \geq 0$ and a vertex set $X \subsetneq V$ with $|X| = k$, we call X a *k-cut in* G if $G - X$ is not connected. A directed graph is called *k-connected* if it contains more than k vertices and no k'-cut for every $k' < k$. A *k-connected component* of G is a maximal k-connected induced subgraph $G[X]$ $(X \subseteq V$ with $|X| \geq k)$.

For vertex sets $X, Y, Z \subseteq V$, we say that X *separates* Y *from* Z *in* G if every two vertices $y \in Y \setminus X$ and $z \in Z \setminus X$ are contained in different connected components of $G - X$. In particular, if X separates Y and Z in G and $Y \setminus X \neq \emptyset \neq Z \setminus X$, then X is an $|X|$-cut in G.

Suppose that G is embedded on a plane. We call a unique unbounded face of G the *outer face* of G, and any other face an *inner face*. For a face F of G, let $\mathrm{bd}(F)$ denote the closed walk (whose end vertices coincide with each other) obtained by walking the boundary of F in an arbitrary direction from an arbitrary vertex on it.

2.1.2 Labels

Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G, and $W = (v_0, e_1, v_1, \ldots, e_l, v_l)$ a walk in G. The *label* $\psi_G(W)$ of W is defined as the ordered product $\psi_G(e_l, v_l) \cdots \psi_G(e_2, v_2) \cdot \psi_G(e_1, v_1)$, where $\psi_G(e_i, v_i) := \psi_G(e_i)$ if $e_i = v_{i-1}v_i$ and $\psi_G(e_i, v_i) := \psi_G(e_i)^{-1}$ if $e_i = v_i v_{i-1}$. Note that, for the reversed walk $\bar{W}$ of W, we have $\psi_G(\bar{W}) = \psi_G(W)^{-1}$. In particular, since an arc uv with label α and an arc vu with label α^{-1} are equivalent, we identify such two arcs. We say that W is *balanced* (or a *zero walk*) if $\psi_G(W) = 1_\Gamma$ and *unbalanced* (or a *non-zero walk*) otherwise, and also that G is *balanced* if G contains no unbalanced cycle. Note that whether a cycle is balanced or not does not depend on the choices of the direction and the end vertex, since $\psi_G(\bar{C}) = \psi_G(C)^{-1}$ and $\psi_G(C') = \psi_G(e_1) \cdot \psi_G(C) \cdot \psi_G(e_1)^{-1}$, where $C = (v_0, e_1, v_1, \ldots, e_l, v_l = v_0)$ and $C' = (v_1, e_2, v_2, \ldots, e_l, v_l = v_0, e_1, v_1)$. Hence, when we consider whether a cycle is balanced or not, we can choose the direction and the end vertex arbitrarily.

For distinct vertices $s, t \in V$, let $l(G; s, t)$ be the set of all possible labels of s–t paths in G. When $l(G; s, t) = \{\alpha\}$ for some $\alpha \in \Gamma$, we also denote the element α itself by $l(G; s, t)$. Without loss of generality, we may assume that there is no vertex $v \in V$ that is not contained in any s–t path, since such a vertex does not make any effect on $l(G; s, t)$. To consider only such cases, let $\mathcal{D}$ be the set of all triplets (G', s, t) such that G' is a Γ-labeled graph with two specified vertices $s, t \in V(G')$ in which every vertex is contained in some s–t path. The following lemma guarantees that one can efficiently obtain a maximal induced subgraph G' of G such that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$ by computing a 2-connected component of a graph (e.g., by [6]).

Lemma 2. *For a Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$, $(G, s, t) \in \mathcal{D}$ if and only if the graph obtained from G by adding a new node $r \notin V$ and two arcs from r to s and from r to t is 2-connected.*

Proof. We may assume that s and t are in the same connected component of G (otherwise, G contains no s–t path). Let Y' be the vertex set of the 2-connected component of $G' := (V + r,\ E \cup \{e_s = rs,\ e_t = rt\})$ that contains both of s and t (such a component exists because of the assumption), and $Y := Y' - r$ (note that Y' must contain r). We show that a vertex $v \in V$ is contained in some s–t path in G if and only if $v \in Y$.

If $v \notin Y$, then G' contains a 1-cut $x \in V - v$ separating v from r. Hence, any r–v path in G' intersects x, and so do any s–v path and any t–v path in G. This implies that G contains no s–t path intersecting v.

If $v \in Y$, then $G'[Y + r]$ contains two r–v paths which do not share their inner vertices by Menger's theorem (see, e.g., [4, Chapter 3]). Each of such paths must intersect either s or t, and hence we can construct an s–t path in G intersecting v by concatenating these two paths and removing r from it. $\square$

2.2 Finding a non-zero path

In this section, we show that a non-zero s–t path can be found (i.e., Problem (II) can be solved) efficiently by using well-known properties of Γ-labeled graphs. The following techniques are often utilized in dealing with Γ-labeled graphs (see, e.g., [2, 3, 18]).

Definition 3 (Shifting)**.** Let $G = (V, E)$ be a Γ-labeled graph. For a vertex $v \in V$ and an element $\alpha \in \Gamma$, *shifting* (*a label function* ψ_G) *by* α *at* v means the following operation: update ψ_G to ψ'_G defined as, for each $e \in E$,

$$\psi'_G(e) := \begin{cases} \psi_G(e) \cdot \alpha^{-1} & (e \in \delta_G(v) \text{ leaves } v), \\ \alpha \cdot \psi_G(e) & (e \in \delta_G(v) \text{ enters } v), \\ \psi_G(e) & (\text{otherwise}). \end{cases}$$

Shifting at $v \in V$ does not change the label of any walk whose end vertices are not v, and neither that of any cycle C whose end vertex is v up to conjugate, i.e., $\psi'_G(C) = \alpha \cdot \psi_G(C) \cdot \alpha^{-1}$. Furthermore, when we apply shifting multiple times, the order of applications does not make any effect on the resulting label function, since each arc is affected only by shifting at its head or tail, which does not interfere with each other. We say that two Γ-labeled graphs G_1 and G_2 are (s, t)-*equivalent* if G_2 is obtained from G_1 by shifting by some $\alpha_v \in \Gamma$ at each $v \in V \setminus \{s, t\}$ (and then G_1 is obtained from G_2 by shifting by α_v^{-1} at each v). Note that $l(G_1; s, t) = l(G_2; s, t)$ if G_1 and G_2 are (s, t)-equivalent.

Lemma 4. *For a connected and balanced Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$, one can find in polynomial time a Γ-labeled graph G' which is (s, t)-equivalent to G such that*

$$\psi_{G'}(e) = \begin{cases} \alpha & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma & (\text{otherwise}), \end{cases}$$

for every arc $e \in E(G') = E$ and for some $\alpha \in \Gamma$ (in fact, $\alpha = l(G; s, t)$).

Proof. Take an arbitrary spanning tree T of G, and assume that all arcs in T are directed toward t. Consider the following procedure. Let $X := \{t\}$. While $X \neq V$, take a neighbor $v \in N_T(X)$, apply shifting the current label function ψ by $\psi(e)$ at v for a unique arc $e \in \delta_T(v) \cap \delta_T(X)$ from v to X (so that $\psi(e) = 1_\Gamma$ after the shifting), and update $X := X + v$. This procedure takes

$O(|E|)$ time, since it just performs breadth first search once and shifting $|V|-1$ times (note that the label of each arc changes at most twice).

After the procedure, we have $\psi(e) = 1_\Gamma$ for every arc $e \in E(T)$, and also for every arc $e \in E$ since G is balanced. Suppose that we applied shifting by α at s. Then, we obtain desired G' by shifting ψ by α^{-1} at s after the procedure. Note that G' is (s,t)-equivalent to G since the resulting label function does not depend on the order of applications of shifting. $\square$

Lemma 5. *For any $(G, s, t) \in \mathcal{D}$, $|l(G; s, t)| = 1$ if and only if G is balanced.*

Proof. "If" part is obvious from Lemma 4. To prove the converse direction, suppose that G is not balanced and $|V(G)| \geq 3$, and let C be an unbalanced cycle in G. Since $(G, s, t) \in \mathcal{D}$ implies that $G + st$ is 2-connected by Lemma 2, for any distinct $x, y \in V(C)$, there exist two disjoint paths (possibly of length 0, i.e., $s = x$ or $y = t$) between $\{s, t\}$ and $\{x, y\}$ in G by Menger's theorem. Take an s–x path P and a y–t path Q in G so that $V(P) \cap V(C) = \{x\}$, $V(Q) \cap V(C) = \{y\}$, and $V(P) \cap V(Q) = \emptyset$, and choose x as the end vertex of C. Since $\psi_G(\bar{C}[x, y])^{-1} \cdot \psi_G(C[x, y]) = \psi_G(C) \neq 1_\Gamma$, we have $\psi_G(C[x, y]) \neq \psi_G(\bar{C}[x, y])$. Hence, by extending $C[x, y]$ and $\bar{C}[x, y]$ using P and Q, we can construct two s–t paths in G whose labels are distinct, which implies $|l(G; s, t)| \geq 2$. $\square$

Lemmas 2, 4, and 5 lead to the following proposition.

Proposition 6. *Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then, for any $\alpha \in \Gamma$, one can test whether $l(G; s, t) \subseteq \{\alpha\}$ or not in polynomial time. Furthermore, if $l(G; s, t) \not\subseteq \{\alpha\}$, then one can find an s–t path P with $\psi_G(P) \neq \alpha$ in polynomial time.*

2.3 New operations

For our characterization of triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 2$, we introduce a few new operations which do not change $l(G; s, t)$. Let $(G = (V, E), s, t) \in \mathcal{D}$, and recall that $G[\![X]\!] := G[X \cup N_G(X)] - E(N_G(X))$ for a vertex set $X \subseteq V$.

Definition 7 (2-contraction)**.** For a vertex set $X \subseteq V \setminus \{s, t\}$ such that $N_G(X) = \{x, y\}$ for some distinct $x, y \in V$ and $G[\![X]\!]$ is connected, the 2-*contraction* of X is the following operation (see Fig. 1):

- remove all vertices in X, and
- add a new arc from x to y with label α for each $\alpha \in l(G[\![X]\!]; x, y)$ if there is no such arc.

The resulting graph is denoted by $G /_2 X$. A vertex set $X \subseteq V \setminus \{s, t\}$ is said to be 2-*contractible in* G if the 2-contraction of X can be performed in G and in particular $G[\![X]\!] \neq G$.

Definition 8 (3-contraction)**.** For a vertex set $X \subseteq V \setminus \{s, t\}$ such that $|N_G(X)| = 3$, $G[X]$ is connected, and $G[\![X]\!]$ is balanced, the 3-*contraction* of X is the following operation (see Fig. 2):

- remove all vertices in X, and
- add a new arc from x to y with label $l(G[\![X]\!]; x, y)$ (which consists of a single element by Lemma 5) for each pair of $x, y \in N_G(X)$ if there is no such arc.

The resulting graph is denoted by $G /_3 X$. A vertex set $X \subseteq V \setminus \{s, t\}$ is said to be 3-*contractible in* G if the 3-contraction of X can be performed in G.

The 2-contraction and the 3-contraction are analogous to the operation which is performed in Condition 3 in Theorem 1, and we use the same term "contraction" to refer to each of them. Any contraction does not change $l(G; s, t)$, since each s–t path cannot enter $G[\![X]\!]$ after leaving it once (i.e., cannot traverse arcs in $G[\![X]\!]$ intermittently). Moreover, we also have $(G', s, t) \in \mathcal{D}$ for the resulting graph G' after any contraction.

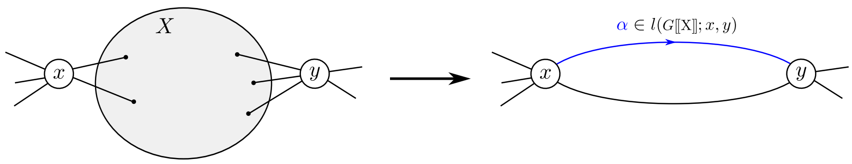

Figure 1: 2-contraction.

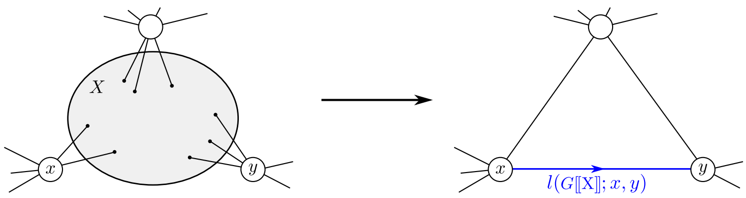

Figure 2: 3-contraction.

3 Main Results

3.1 Algorithmic results

As described in Section 2.2, Problem (II) can be solved efficiently, i.e., one can find a non-zero s–t path in polynomial time (Proposition 6). The following theorem, one of our main results, is the first nontrivial extension of this property, which claims that not only one label but also another can be forbidden simultaneously.

Theorem 9. *Let $G = (V, E)$ be a Γ-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then, for any distinct $\alpha, \beta \in \Gamma$, one can test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not in polynomial time. Furthermore, if $l(G; s, t) \not\subseteq \{\alpha, \beta\}$, then one can find an s–t path P with $\psi_G(P) \notin \{\alpha, \beta\}$ in polynomial time.*

Such an algorithm is constructed based on characterizations of Γ-labeled graphs with exactly two possible labels of s–t paths, which are shown in Section 3.2. Our algorithm and a proof of this theorem are presented in Section 4. It should be mentioned that this theorem leads to a solution to Problem (I) for $\Gamma \simeq \mathbb{Z}_3$.

Corollary 10. *Let $G = (V, E)$ be a $\mathbb{Z}_3$-labeled graph with a label function ψ_G and two specified vertices $s, t \in V$. Then one can compute $l(G; s, t)$ in polynomial time. Furthermore, for each $\alpha \in l(G; s, t)$, one can find an s–t path P with $\psi_G(P) = \alpha$ in polynomial time.*

3.2 Characterizations

Recall that $\mathcal{D}$ denotes the set of all triplets (G, s, t) such that G is a Γ-labeled graph with $s, t \in V(G)$ in which every vertex is contained in some s–t path. In this section, we provide a complete characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. We consider two cases separately: when $\alpha\beta^{-1} = \beta\alpha^{-1}$ and when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$.

First, we give a characterization in the easier case: when $\alpha\beta^{-1} = \beta\alpha^{-1}$. Note that this case does not appear when $\Gamma \simeq \mathbb{Z}_3$. The following proposition holds analogously to Lemmas 4 and 5 in Section 2.2, which characterize triplets $(G, s, t) \in \mathcal{D}$ with $|l(G; s, t)| = 1$.

Proposition 11. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} = \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if G is not balanced and there exists a Γ-labeled graph G' which is (s, t)-equivalent to G such that*

$$\psi_{G'}(e) = \begin{cases} \alpha \text{ or } \beta & (e \in \delta_G(s) \text{ leaves } s), \\ \alpha^{-1} \text{ or } \beta^{-1} & (e \in \delta_G(s) \text{ enters } s), \\ 1_\Gamma \text{ or } \alpha\beta^{-1} & (\text{otherwise}), \end{cases} \tag{$*$}$$

for every arc $e \in E(G') = E(G)$. Moreover, one can find such G' in polynomial time if exists.

Proof. "If" part is easy to see as follows. Since G is not balanced, $|l(G; s, t)| \geq 2$ by Lemma 5. Furthermore, since $\alpha\beta^{-1} = \beta\alpha^{-1}$, the label of any s–t path in G' is α or β. Hence, the (s, t)-equivalence between G and G' leads to $l(G; s, t) = l(G'; s, t) = \{\alpha, \beta\}$.

The converse direction is rather difficult. Similarly to the proof of Lemma 4, take an arbitrary spanning tree T of G and apply shifting at each $v \in V - t$ so that $\psi(e) = 1_\Gamma$ for every arc $e \in E(T)$, where ψ denotes the resulting label function. Since $l(G; s, t) = \{\alpha, \beta\}$ and $l(T; s, t) = 1_\Gamma$, we applied shifting by α or β at s. Hence, by shifting ψ by α^{-1} or β^{-1}, respectively, at s after the above procedure, we can obtain a Γ-labeled graph G' which is (s, t)-equivalent to G, and this G' is in fact desired one.

To see this, suppose to the contrary that some arc $e' \in E(G')$ does not satisfy $(*)$, and let $E' \subsetneq E(G')$ be the set of arcs satisfying $(*)$. Note that $E(T) \subseteq E'$, and hence $G'[E']$ is connected. Take an s–t path P in G' with $E(P) \setminus E' \neq \emptyset$ so that $|E(P) \setminus E'|$ is minimized.

If $|E(P) \setminus E'| = 1$, then $\psi_{G'}(P) \notin \{\alpha, \beta\}$, which contradicts $l(G'; s, t) = l(G; s, t) = \{\alpha, \beta\}$. Otherwise, we have $|E(P) \setminus E'| \geq 2$. Let $e_1, e_2 \in E(P) \setminus E'$ be the first two such arcs traversed in walking along P, and Q the subpath of P connecting e_1 and e_2 (hence, $E(Q) \subseteq E'$). Since $G'[E']$ is connected, there exists a path R from $u \in V(Q)$ to $w \in V(P) \setminus V(Q)$ in $G'[E']$. We can construct an s–t path P' from P by replacing $P[u, w]$ (or $P[w, u]$) with R (or $\bar{R}$) such that $\emptyset \neq E(P') \setminus E' \subsetneq E(P) \setminus E'$ (since $|E(P') \cap \{e_1, e_2\}| = 1$). This implies that $1 \leq |E(P') \setminus E'| \leq |E(P) \setminus E'| - 1$, which contradicts the choice of R. $\square$

We next discuss the main case, which is much more difficult: when $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. The following theorem, one of our main results, completes a characterization of triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ for some distinct $\alpha, \beta \in \Gamma$. The definition of the set $\mathcal{D}_{\alpha,\beta} \subseteq \mathcal{D}$, which appears in the theorem, is shown later through Definitions 13–15 in Section 3.3. In short, $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ if G is constructed by "gluing" together "nice" planar Γ-labeled graphs (and some trivial Γ-labeled graphs) and their derivations.

Theorem 12. *Let α and β be distinct elements in Γ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. For any $(G, s, t) \in \mathcal{D}$, $l(G; s, t) = \{\alpha, \beta\}$ if and only if $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$.*

Recall that $|l(G; s, t)| = 1$ if and only if G is balanced by Lemma 5, which can be easily tested by Lemma 4. Hence, these characterizations lead to the first nontrivial classification of Γ-labeled graphs in terms of the number of possible labels of s–t paths, and the classification is also complete when $\Gamma \simeq \mathbb{Z}_3$.

3.3 Definition of $\mathcal{D}_{\alpha,\beta}$

Fix distinct elements $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. To characterize triplets $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$, let us define several sets of triplets $(G, s, t) \in \mathcal{D}$ for which it is easy to see that $l(G; s, t) = \{\alpha, \beta\}$. Theorem 12 claims that any triplet $(G, s, t) \in \mathcal{D}$ with $l(G; s, t) = \{\alpha, \beta\}$ is in fact contained in one of them.

Definition 13. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, let $\mathcal{D}^0_{\alpha,\beta}$ be the set of all triplets $(G, s, t) \in \mathcal{D}$ satisfying one of the following conditions.

(A) There exists a Γ-labeled graph G' which is not balanced and is (s, t)-equivalent to G such that either

- the label of every arc in $G' - s$ is 1_Γ and in $\delta_{G'}(s)$ is α or β, where all arcs in $\delta_{G'}(s)$ are assumed to leave s (see Fig. 3), or
- the label of every arc in $G' - t$ is 1_Γ and in $\delta_{G'}(t)$ is α or β, where all arcs in $\delta_{G'}(t)$ are assumed to enter t (see Fig. 4).

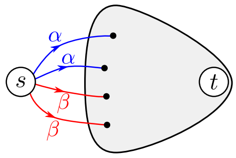

Figure 3: The former of Case (A).

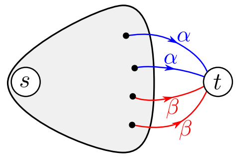

Figure 4: The latter of Case (A).

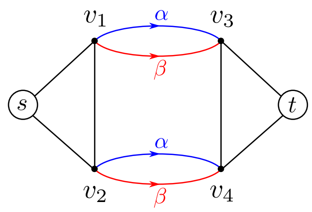

Figure 5: Case (B).

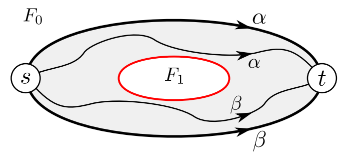

Figure 6: Case (C).

(B) G is (s,t)-equivalent to the Γ-labeled graph which consists of six vertices s, v_1, v_2, v_3, v_4, t, six arcs $sv_1, sv_2, v_1v_2, v_3v_4, v_3t, v_4t$ with label 1_Γ, and two pairs of two parallel arcs from v_i to v_{i+2} $(i = 1, 2)$ whose labels are both α and β (see Fig. 5).

(C) G can be embedded on a plane with the face set $\mathcal{F}$ satisfying the following conditions (see Fig. 6):

- both s and t are on the boundary of the outer face $F_0 \in \mathcal{F}$,
- one s–t path along $\mathrm{bd}(F_0)$ is of label α and the other is of β, and
- there exists a unique inner face F_1 whose boundary is unbalanced, i.e., $\psi_G(\mathrm{bd}(F_1)) \neq 1_\Gamma$ and $\psi_G(\mathrm{bd}(F)) = 1_\Gamma$ for any $F \in \mathcal{F} \setminus \{F_0, F_1\}$.

It is not difficult to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$.

For the following definitions, recall the operations called the "contractions," which are defined in Section 2.3 (see Definitions 7 and 8).

Definition 14. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, we define $\mathcal{D}^1_{\alpha,\beta}$ as the minimal set of triplets $(G, s, t) \in \mathcal{D}$ with the following conditions:

- $\mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}^1_{\alpha,\beta}$, and
- if $(G/_3 X, s, t) \in \mathcal{D}^1_{\alpha,\beta}$ for some 3-contractible $X \subseteq V \setminus \{s, t\}$, then $(G, s, t) \in \mathcal{D}^1_{\alpha,\beta}$.

We are now ready to define $\mathcal{D}_{\alpha,\beta}$.

Definition 15. For distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, we define $\mathcal{D}_{\alpha,\beta}$ as the minimal set of triplets $(G, s, t) \in \mathcal{D}$ with the following conditions:

- $\mathcal{D}^1_{\alpha,\beta} \subseteq \mathcal{D}_{\alpha,\beta}$, and
- if $(G/_2 X, s, t) \in \mathcal{D}_{\alpha,\beta}$ for some $X \subseteq V \setminus \{s, t\}$ such that either $G[\![X]\!]$ is balanced or $(G[\![X]\!], x, y) \in \mathcal{D}^1_{\alpha',\beta'}$, where $N_G(X) = \{x, y\}$ and $\alpha', \beta' \in \Gamma$ satisfy $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, then $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$.

Note that the first condition can be replaced with $(G_0, s, t) \in \mathcal{D}_{\alpha,\beta}$, where G_0 consists of two parallel arcs from s to t whose labels are α and β.

It is easy to see that $l(G; s, t) = \{\alpha, \beta\}$ for any triplet $(G, s, t) \in \mathcal{D}_{\alpha,\beta}$ since any contraction does not change $l(G; s, t)$. A proof of the non-trivial direction ("only if" part of Theorem 12) is presented later in Section 5.

4 Algorithm

In this section, we give a proof of Theorem 9. That is, we present a polynomial-time algorithm to test whether $l(G; s, t) \subseteq \{\alpha, \beta\}$ or not for given distinct $\alpha, \beta \in \Gamma$ and to find an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ if $l(G; s, t) \not\subseteq \{\alpha, \beta\}$, in a given Γ-labeled graph $G = (V, E)$ with $s, t \in V$. It should be mentioned that, when $\Gamma \simeq \mathbb{Z}_3$, such an algorithm can compute $l(G; s, t)$ itself and find an s–t path of label α for each $\alpha \in l(G; s, t)$. Without loss of generality, we assume that G does not have parallel arcs with the same label.

4.1 Algorithm description

For the simple description, we separate our algorithm into two parts: to test whether $|l(G; s, t)| \leq 2$ or not and return at most two s–t paths which attain all labels in $l(G; s, t)$ when $|l(G; s, t)| \leq 2$, and to find three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$

We first present the former algorithm. Note again that this algorithm can compute $l(G; s, t)$ itself when $\Gamma \simeq \mathbb{Z}_3$. Throughout this algorithm, let $G' = (V', E')$ denote a temporary Γ-labeled graph currently considered.

TestTwoLabels(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$.

Output The set $l(G; s, t)$ of all possible labels of s–t paths in G with those which attain the labels if $|l(G; s, t)| \leq 2$, and "$|l(G; s, t)| \geq 3$" otherwise.

Step 0. Compute the set X of vertices which are not contained in any s–t path in G by Lemma 2. If $X = V$, then halt with returning $\emptyset$ since there is no s–t path in G. Otherwise, set $G' \leftarrow G - X$. Note that $(G', s, t) \in \mathcal{D}$ and $l(G'; s, t) = l(G; s, t)$.

Step 1. Test whether G' is balanced or not by Lemma 4 (i.e., take an arbitrary spanning tree, and apply shifting along it). If G' is balanced, then halt with returning the label of an arbitrary s–t path in G with the path. Otherwise, by using an unbalanced cycle, obtain two s–t paths in G whose labels are distinct (cf. the proof of Lemma 5), say $\alpha, \beta \in \Gamma$. In the following steps, we check whether $l(G'; s, t) = \{\alpha, \beta\}$ or not.

Step 2. If $\alpha\beta^{-1} = \beta\alpha^{-1}$, then check the condition in Proposition 11. Return $\{\alpha, \beta\}$ with the two s–t paths in G obtained in Step 1 if it is satisfied, and "$|l(G; s, t)| \geq 3$" otherwise. Otherwise (i.e., if $\alpha\beta^{-1} \neq \beta\alpha^{-1}$), to make G' 2-connected (unless $V' = \{s, t\}$), add to G' a new arc from s to t with label α (or β) if s and t are not adjacent in G'.

Step 3. While G' is not 3-connected and $|V'| \geq 4$, do the following procedure. Let $\{x, y\} \subsetneq V'$ be a 2-cut in G', and X the vertex set of a connected component of $G' - \{x, y\}$ with $X \cap \{s, t\} = \emptyset$ (such X exists, since s and t are adjacent in G'). Test whether $|l(G'[\![X]\!]; x, y)| \leq 2$ or not recursively by TestTwoLabels$(G'[\![X]\!], x, y)$. Update $G' \leftarrow G'/_2X$ (2-contraction) if $|l(G'[\![X]\!]; x, y)| \leq 2$, and return "$|l(G; s, t)| \geq 3$" otherwise.

Step 4. While there exists a 3-contractible vertex set $X \subseteq V' \setminus \{s, t\}$, update $G' \leftarrow G'/_3X$ (3-contraction). Note that here we use Lemma 4.

Step 5. If $|V'| \leq 6$, then compute $l(G', s, t)$ by enumerating all s–t paths in G' and return the result. Otherwise, test whether $(G', s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not by Lemma 16. Return $\{\alpha, \beta\}$ with the s–t paths in G obtained in Step 1 if $(G', s, t) \in \mathcal{D}^0_{\alpha,\beta}$, and "$|l(G; s, t)| \geq 3$" otherwise.

Next, we show the latter algorithm, which finds three s–t paths whose labels are distinct when it has turned out that $|l(G; s, t)| \geq 3$. Also note again that this algorithm finds three s–t paths which attain all labels when $\Gamma \simeq \mathbb{Z}_3$.

FindThreePaths(G, s, t)

Input A Γ-labeled graph $G = (V, E)$ and distinct vertices $s, t \in V$ such that $|l(G; s, t)| \geq 3$.

Output Three s–t paths in G whose labels are distinct.

Step 0. If $V = \{s, t\}$, then halt with returning three s–t paths each of which consists of a single arc from s to t in E. Note that E must contain at least three such parallel arcs with distinct labels.

Step 1. For each $s' \in N_G(s) - t$, test whether $|l(G - s; s', t)| \leq 2$ or not by TestTwoLabels$(G - s, s', t)$.

Step 2. If $|l(G - s; s', t)| \leq 2$ for all $s' \in N_G(s) - t$, then we have already obtained s'–t paths which attain all labels in $l(G - s; s', t)$. Choose three s–t paths whose labels are distinct among the s–t paths obtained by extending such s'–t paths using an arc (possibly parallel arcs) $ss' \in E$ for each $s' \in N_G(s) - t$ and the s–t paths each of which consists of a single arc $st \in E$, and halt with returning them.

Step 3. Otherwise, for at least one $\tilde{s} \in N_G(s) - t$, we obtained $|l(G - s; \tilde{s}, t)| \geq 3$. Then, recursively by FindThreePaths$(G - s, \tilde{s}, t)$, find three $\tilde{s}$–t paths whose labels are distinct. Extend the three $\tilde{s}$–t paths using an arc $s\tilde{s} \in E$, and return the extended s–t paths.

4.2 Proof of Theorem 9

Before starting the proof, we show the detailed procedure of Step 5 in TestTwoLabels.

Lemma 16. *Let $(G, s, t) \in \mathcal{D}$. Suppose that $G = (V, E)$ is 3-connected and contains no 3-contractible vertex set, $|V| > 6$, s and t are adjacent, and $\{\alpha, \beta\} \subseteq l(G; s, t)$ for some distinct $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$. Then, one can test whether $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not in polynomial time.*

Proof. Since $|V| > 6$, it is not necessary to consider Case (B) in Definition 13. Besides, Case (A) is easily checked by testing whether $G - s$ or $G - t$ is balanced or not. Hence, in what follows, we assume that (G, s, t) is not in Case (A) or (B) and focus on Case (C).

First, test the planarity of G. If G is not planar, then we can conclude $(G, s, t) \notin \mathcal{D}^0_{\alpha,\beta}$. Otherwise, compute an embedding of G on a plane in which both s and t are on the outer boundary (because of an arc $st \in E$, there exists a face on whose boundary both s and t are). It should be noted that such a planar embedding can be computed in polynomial time (e.g., by [7]). Since G is 3-connected, the face set is unique if there are no parallel arcs (see, e.g., [4, Chapter 4]). Although there may be parallel arcs in G, we can say that the number of parallel arcs is bounded as seen below.

Claim. We may assume that there is no parallel arcs between s and t.

Suppose that there exist parallel arcs from s to t, which may be assumed to have distinct labels. Moreover, we may assume that there are exactly two such arcs $e_\alpha, e_\beta \in E$ with labels α, β, respectively, since otherwise, we have $|l(G; s, t)| \geq 3$ and hence we can conclude $(G, s, t) \notin \mathcal{D}^0_{\alpha,\beta}$. Since $|V| > 6$ and $(G, s, t) \in \mathcal{D}$, there exists an s–t path in $G - \{e_\alpha, e_\beta\}$, and let γ be its label. If $\alpha \neq \gamma \neq \beta$, then $|l(G; s, t)| \geq 3$. Otherwise, remove e_γ from G. Note that this removal does not violate the hypotheses of this lemma, and does not make any effect on whether $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ or not.

Claim. We may assume that there exists at most one pair of parallel arcs.

Suppose that there exist parallel arcs from x to y with distinct labels, where $\{x, y\} \neq \{s, t\}$. Then, by the 3-connectivity of G, the parallel arcs form an inner face whose boundary is unbalanced (since otherwise $\{x, y\}$ is a 2-cut in G). Hence, there is a unique pair of such parallel arcs if $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$, since the existence of at least two pairs of parallel arcs immediately implies that there exist at least two inner faces whose boundaries are unbalanced.

Recall that we have to test whether there exists an embedding of G such that the outer boundary is unbalanced and there exists a unique inner face whose boundary is unbalanced. Since a pair of parallel arcs is unique if exists, there are at most two possible face sets of G. Furthermore, since there exists exactly one arc from s to t, both of the two faces whose boundaries share the arc $st \in E$ can be the outer face, i.e., there are two choices of the outer face. It can be done in polynomial time to check, in each of the at most four ($= 2 \times 2$) cases, whether exactly one inner face has an unbalanced boundary or not, and hence one can do the whole procedure in polynomial time. □

We are now ready to prove Theorem 9.

Proof of Theorem 9. Recall that our goal is to test whether $|l(G; s, t)| \leq 2$ or not, and to find $\min\{3, |l(G; s, t)|\}$ s–t paths whose labels are distinct. These are achieved as follows. For the input triplet (G, s, t) (which may not be in $\mathcal{D}$), we first test whether $|l(G; s, t)| \leq 2$ or not by TestTwoLabels(G, s, t). If we obtain $|l(G; s, t)| \leq 2$, then we also obtain at most two s–t paths in G which attain all labels in $l(G; s, t)$. Otherwise, we can obtain three s–t paths whose labels are distinct by FindThreePaths(G, s, t). Hence, it suffices to show the correctness and polynomiality of these two algorithms.

The correctness of these two algorithms is almost obvious. It should be noted that we have $l(G'; s, t) = l(G; s, t)$ and $(G', s, t) \in \mathcal{D}$ at any step of TestTwoLabels(G, s, t). This follows from the fact that the 2-contractions in Step 3 and the 3-contractions in Step 4 do not change $l(G'; s, t)$ or violate $(G', s, t) \in \mathcal{D}$.

We finally confirm the polynomiality of the two algorithms. Let $T_{\mathrm{labels}}(n)$ and $T_{\mathrm{paths}}(n)$ denote the computational time of TestTwoLabels(G, s, t) and FindThreePaths(G, s, t), respectively, where n is the number of vertices in G. It is easy to see that TestTwoLabels runs in polynomial time, i.e., $T_{\mathrm{labels}}(n)$ is polynomially bounded. Note that, in the recursion step (Step 3), we just divide the graph G' into two smaller graphs which have $|V'| - |X|$ and $|X| + 2$ vertices, and in the 3-contraction step (Step 4), it suffices to check all 3-cuts in G', whose number is $O(n^3)$. For FindThreePaths, by a recurrence relation

$$T_{\mathrm{paths}}(n) \leq n \cdot T_{\mathrm{labels}}(n-1) + T_{\mathrm{paths}}(n-1) + \mathrm{poly}(n),$$

we have $T_{\mathrm{paths}}(n) \leq n^2 \cdot T_{\mathrm{labels}}(n) + \mathrm{poly}(n)$. Hence, $T_{\mathrm{paths}}(n)$ is also polynomially bounded. □

5 Proof of Necessity Part of Theorem 12

In this section, we give a proof of the necessity part of Theorem 12, and begin with its sketch.

5.1 Proof sketch

To derive a contradiction, assume that there exist distinct $\alpha, \beta \in \Gamma$ and a triplet $(G, s, t) \in \mathcal{D}$ such that $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, $l(G; s, t) = \{\alpha, \beta\}$, and $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$. We choose such $\alpha, \beta \in \Gamma$ and $(G, s, t) \in \mathcal{D}$ so that G is as small as possible.

Fix an arbitrary arc e_0 in G leaving s, and consider the graph $G' := G - e_0$. By using the minimality of G, we can show that $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$ (cf. Claims 23 and 24). We consider the following two cases separately: when $(G', s, t) \in \mathcal{D}^1_{\alpha,\beta}$ and when not (Sections 5.4 and 5.5, respectively).

In both cases, we can embed a graph $\tilde{G}$ obtained from G' by at most one 3-contraction on a plane so that the conditions of Case (C) in Definition 13 are satisfied (or derive a contradiction).

By expanding a vertex set and adding e_0, we try to extend the planar embedding of $\tilde{G}$ to G. Then, we have one of the following cases.

- Such an extension is possible, i.e., G can be embedded on a plane with the conditions of Case (C) in Definition 13. This contradicts that $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$.
- G contains a contractible vertex set, which contradicts that G is a minimal counterexample (cf. Claims 21 and 22).
- We can construct an s–t path of label $\gamma \in \Gamma \setminus \{\alpha, \beta\}$ in G by using e_0 and some arcs in G', which contradicts that $l(G; s, t) = \{\alpha, \beta\}$.

In each case, we have a contradiction, which completes the proof. We note that Theorem 1 plays an important role in this case analysis.

5.2 Useful lemmas

Before starting the proof, we show several lemmas which are utilized in it. Fix distinct elements $\alpha, \beta \in \Gamma$ with $\alpha\beta^{-1} \neq \beta\alpha^{-1}$.

Lemma 17. *For any $(G = (V, E), s, t) \in \mathcal{D}_{\alpha,\beta}$, we have the following properties.*

(1) *Let G' be the graph obtained from G by shifting by $\gamma \in \Gamma$ at s. Then, $(G', s, t) \in \mathcal{D}_{\alpha',\beta'}$, where $\alpha' := \alpha\gamma^{-1}$ and $\beta' := \beta\gamma^{-1}$.*

(2) *Let $G' := (V + s', E + e')$ be the graph obtained from G by adding a new vertex s' and a new arc $e' = s's$ with label $\gamma \in \Gamma$. Then, $(G', s', t) \in \mathcal{D}_{\alpha',\beta'}$, where $\alpha' := \alpha\gamma$ and $\beta' := \beta\gamma$.*

(3) *If $G = G' /_2 X$ for a Γ-labeled graph G' and $X \subseteq V(G') \setminus \{s, t\}$ with $(G'[\![X]\!], x, y) \in \mathcal{D}_{\alpha',\beta'}$, where $N_{G'}(X) = \{x, y\}$ and $\alpha', \beta' \in \Gamma$ satisfy $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, then $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$.*

Proof. (1) We first confirm that, if $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$, then $(G', s, t) \in \mathcal{D}^0_{\alpha',\beta'}$. The former of Case (A) and Case (C) are obvious (cf. Definition 13). In the latter of Case (A), apply shifting by γ at each $v \in V \setminus \{s, t\}$, and in Case (B), do so at v_1 and v_2.

We next show that, if $(G, s, t) \in \mathcal{D}^1_{\alpha,\beta}$, then $(G', s, t) \in \mathcal{D}^1_{\alpha',\beta'}$. Suppose that $(G, s, t) \in \mathcal{D}^1_{\alpha,\beta}$. Then, one can obtain a Γ-labeled graph $\tilde{G}$ such that $(\tilde{G}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ from G by applying 3-contractions. Since any shifting does not make effect on whether a Γ-labeled graph is balanced or not, the same 3-contractions can be applied to G', and we obtain a Γ-labeled graph $\tilde{G}'$ such that $(\tilde{G}', s, t) \in \mathcal{D}^0_{\alpha',\beta'}$ as a result. Thus we have done.

By the definition of $\mathcal{D}_{\alpha,\beta}$ (Definition 15), there exists a sequence $G_0, G_1, \ldots, G_r$ of Γ-labeled graphs satisfying the following conditions:

- $G_r = G$,
- G_0 consists of two vertices s and t and two parallel arcs e_α, e_β from s to t whose labels are α and β, respectively, and
- $G_{i-1} = G_i /_2 X_i$ for some $X_i \subseteq V(G_i) \setminus \{s, t\}$ such that either $G_i[\![X_i]\!]$ is balanced or $(G_i[\![X_i]\!], x_i, y_i) \in \mathcal{D}^1_{\alpha_i,\beta_i}$, where $N_{G_i}(X_i) = \{x_i, y_i\}$ and $\alpha_i, \beta_i \in \Gamma$ satisfy $\alpha_i\beta_i^{-1} \neq \beta_i\alpha_i^{-1}$, for each $i = 1, 2, \ldots, r$.

We prove that the same 2-contractions can be applied to G'.

Define $G'_r := G'$. Then, we can inductively construct a Γ-labeled graph $G'_{i-1} := G'_i /_2 X_i$, which coincides with the one obtained from G_{i-1} by shifting by γ at s. This means that we finally obtain a Γ-labeled graph G'_0 from G' by the 2-contractions of X_i ($i = r, r-1, \ldots, 1$), which satisfies $(G'_0, s, t) \in \mathcal{D}^0_{\alpha',\beta'}$ (in Cases (A) and (C)). Thus we have $(G', s, t) \in \mathcal{D}_{\alpha',\beta'}$, since either $G'_i[\![X_i]\!]$ is balanced or $(G'_i[\![X_i]\!], x_i, y_i) \in \mathcal{D}^1_{\alpha'_i,\beta'_i}$, where $\alpha'_i = \alpha_i$ and $\beta'_i = \beta_i$ if $s \notin \{x_i, y_i\}$, and $\alpha'_i = \alpha_i\gamma^{-1}$ and $\beta'_i = \beta_i\gamma^{-1}$ otherwise (assume $x_i = s$ without loss of generality by the symmetry of x_i and y_i).

(2) Similarly to the proof of (1), there exists a sequence $G_0, G_1, \ldots, G_r = G$ such that $G_0 = (\{s,t\}, \{e_\alpha, e_\beta\})$, and G_{i-1} is obtained from G_i by some appropriate 2-contraction. The same 2-contractions can be applied to G', and we obtain the Γ-labeled graph $G'_0 = (\{s', s, t\}, \{e', e_\alpha, e_\beta\})$, which satisfies $(G'_0, s', t) \in \mathcal{D}^0_{\alpha',\beta'}$ (in Cases (A) and (C)). Thus we have $(G', s, t) \in \mathcal{D}_{\alpha',\beta'}$.

(3) Similarly, there exists a sequence $H_0, H_1, \ldots, H_r = G'[\![X]\!]$ such that H_0 consists of two parallel arcs from x to y whose labels are α' and β', and H_{i-1} is obtained from H_i by some appropriate 2-contraction. The same 2-contractions can be applied to G', and we obtain G. This implies that $(G', s, t) \in \mathcal{D}_{\alpha,\beta}$. □

By Lemma 17-(1), it suffices to consider the case when $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$). The following lemma gives a useful characterization of $\mathcal{D}^0_{1_\Gamma,\alpha}$ in Case (C) (cf. Definition 13).

Lemma 18. *Suppose that $\alpha^{-1} \neq \alpha \in \Gamma$ (i.e., $\alpha^2 \neq 1_\Gamma$). For any triplet $(G = (V, E), s, t) \in \mathcal{D}^0_{1_\Gamma,\alpha}$ in Case (C) in Definition 13, there exists a Γ-labeled graph G' which is (s,t)-equivalent to G and embeddable with the following conditions (see Fig. 7).*

1. *The arc set E is partitioned into E^0 and E^1 (i.e., $E^0 \cup E^1 = E$ and $E^0 \cap E^1 = \emptyset$), where $E^i := \{\, e \in E \mid \psi_{G'}(e) = \alpha^i \,\}$ $(i = 0, 1)$.*
2. *There exists an s–t path $P = (s = u_0, e_1, u_1, \ldots, e_l, u_l = t)$ along the outer boundary of $G' - E^1$ such that*
   - *every arc in E^1 is embedded on the outer face of $G' - E^1$ and is from $u_i \in V(P)$ to $u_j \in V(P)$ for some $i < j$, and*
   - *for any distinct arcs $e_1 = u_{i_1}u_{j_1}$, $e_2 = u_{i_2}u_{j_2} \in E^1$, one of two paths $P[u_{i_1}, u_{j_1}]$ and $P[u_{i_2}, u_{j_2}]$ is a subpath of the other.*

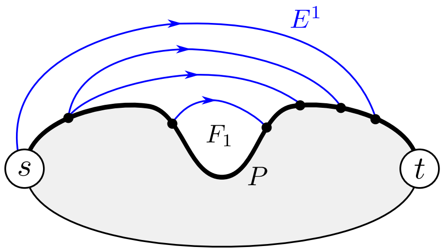

Figure 7: An s–t equivalent embedding of $(G, s, t) \in \mathcal{D}^0_{1_\Gamma,\alpha}$ in Case (C).

Proof. Fix an embedding of G with the conditions of Case (C), and let P_0 and P_1 be the s–t paths along the boundary of the outer face F_0 of G whose labels are 1_Γ and α, respectively.

Let G^* be the dual graph of G (as an undirected graph), i.e., the vertex set of G^* is the face set $\mathcal{F}$ of G, the edge set of G^* coincides with the arc set of G, and each two faces whose boundaries share an arc $e \in E$ in G are connected by the same-named edge e in G^*. Take a shortest F_1–F_0 path Q in $G^* - E(P_0)$. We prove that the second condition holds with $E^1 = E(Q)$.

Note that $G'' := G - E(Q)$ is connected since Q is a shortest path without the corresponding edge to any arc in $E(P_0)$, and that G'' is balanced since F_1 is the unique unbalanced inner face. We then have $l(G''; s, t) = 1_\Gamma$ by Lemma 5. Hence, by Lemma 4, we may assume that $\psi_G(e) = 1_\Gamma$ for every arc $e \in E(G'')$ by shifting at some vertices $v \in V \setminus \{s, t\}$. Thus we obtain G' with the second condition, since $\psi_G(\mathrm{bd}(F)) = 1_\Gamma$ for any $F \in \mathcal{F} \setminus \{F_0, F_1\}$. □

The following two lemmas are utilized to derive a contradiction by constructing an s–t path of label $\gamma \notin \Gamma \setminus \{\alpha, \beta\}$ in G, where $(G, s, t) \in \mathcal{D}$.

Lemma 19. *For a triplet $(G, s, t) \in \mathcal{D}$, if G contains an unbalanced cycle C with $\psi_G(\bar{C}) = \psi_G(C)$, then there exist distinct elements $\alpha', \beta' \in l(G; s, t)$ with $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$.*

Proof. We first note that the equality $\psi_G(\bar{C}) = \psi_G(C)$ does not depend on the choices of the direction and the end vertex of the cycle C. Suppose that G contains such an unbalanced cycle C. By Menger's theorem (cf. the proof of Lemma 5), for some distinct vertices $x, y \in V(C)$, take an s–x path P and a y–t path Q in G so that $V(P) \cap V(C) = \{x\}$, $V(Q) \cap V(C) = \{y\}$, and $V(P) \cap V(Q) = \emptyset$, and choose y as the end vertex of C.

Let $\alpha'' := \psi_G(C[x, y])$ and $\beta'' := \psi_G(\bar{C}[x, y])$, which are distinct since C is unbalanced. We then have $\alpha''\beta''^{-1} = \psi_G(C) = \psi_G(\bar{C}) = \beta''\alpha''^{-1}$. By extending $C[x, y]$ and $\bar{C}[x, y]$ using P and Q, we obtain two s–t paths in G whose labels are $\alpha' := \psi_G(Q) \cdot \alpha'' \cdot \psi_G(P)$ and $\beta' := \psi_G(Q) \cdot \beta'' \cdot \psi_G(P)$, which are distinct. Since $\alpha''\beta''^{-1} = \beta''\alpha''^{-1}$, we have $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$. $\square$

In particular, G contains no unbalanced cycle C with $\psi_G(\bar{C}) = \psi_G(C)$ if $l(G; s, t) = \{\alpha, \beta\}$ (recall that $\alpha\beta^{-1} \neq \beta\alpha^{-1}$) and $(G, s, t) \in \mathcal{D}$.

Lemma 20. *For a triplet $(G, s, t) \in \mathcal{D}$, if there exist two paths P_i $(i = 1, 2)$ in G with the following conditions (see Fig. 8), then $|l(G; s, t)| \geq 3$:*

- *P_i is from s to $x_i \in V \setminus \{s, t\}$ for $i = 1, 2$,*
- *$\psi_G(P_1) \neq \psi_G(P_2)$, and*
- *$\{\alpha', \beta'\} \subseteq l(G - (V(P_i) - x_i); x_i, t)$ for $i = 1, 2$, for some $\alpha', \beta' \in \Gamma$ with $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$.*

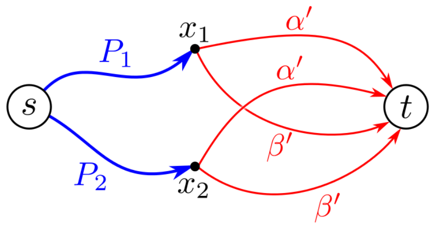

Figure 8: Combination of two labels leads to at least three labels.

Proof. For each $i = 1, 2$, by concatenating P_i and each of two x_i–t paths in $G - (V(P_i) - x_i)$ whose labels are α' and β', we construct four s–t paths whose labels are $\gamma_1 := \alpha' \cdot \psi_G(P_1)$, $\gamma_2 := \beta' \cdot \psi_G(P_1)$, $\gamma_3 := \alpha' \cdot \psi_G(P_2)$, and $\gamma_4 := \beta' \cdot \psi_G(P_2)$.

Suppose to the contrary that $|l(G; s, t)| \leq 2$. Since $\gamma_1 \neq \gamma_2 \neq \gamma_4 \neq \gamma_3 \neq \gamma_1$, we must have $\gamma_1 = \gamma_4$ and $\gamma_2 = \gamma_3$. Hence, $\psi_G(P_1) = \alpha'^{-1} \cdot \beta' \cdot \psi_G(P_2)$ and $\psi_G(P_1) = \beta'^{-1} \cdot \alpha' \cdot \psi_G(P_2)$, which implies $\alpha'^{-1}\beta' = \beta'^{-1}\alpha'$. This is equivalent to $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$, a contradiction. $\square$

5.3 Minimal counterexample

Here, we start a proof of "only if" part of Theorem 12. To derive a contradiction, suppose to the contrary that there exist distinct elements $\alpha, \beta \in \Gamma$ and a triplet $(G, s, t) \in \mathcal{D}$ such that $\alpha\beta^{-1} \neq \beta\alpha^{-1}$, $l(G; s, t) = \{\alpha, \beta\}$, and $(G, s, t) \notin \mathcal{D}_{\alpha,\beta}$. We choose such $\alpha, \beta \in \Gamma$ and $(G = (V, E), s, t) \in \mathcal{D}$ so that the value of $|V| + |E|$ is minimized. Note that we have $|V| \geq 3$ obviously, and we may assume $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$) by Lemma 17-(1). By the minimality, G contains no contractible vertex set as follows.

Claim 21. *There is no 2-contractible vertex set in G.*

Proof. Suppose to the contrary that G contains a 2-contractible vertex set $X \subseteq V \setminus \{s,t\}$ with $N_G(X) = \{x,y\}$. Since $(G,s,t) \in \mathcal{D}$, we also have $(G[\![X]\!],x,y) \in \mathcal{D}$, where recall that $G[\![X]\!] := G[X \cup N_G(X)] - E(N_G(X))$. If $|l(G[\![X]\!];x,y)| \geq 3$, then we also have $|l(G;s,t)| \geq 3$ (since G contains two disjoint paths between $\{s,t\}$ and $\{x,y\}$ by Lemma 2 and Menger's Theorem), a contradiction. In the case that $l(G[\![X]\!];x,y) = \{\alpha',\beta'\}$ for distinct $\alpha',\beta' \in \Gamma$ with $\alpha'\beta'^{-1} = \beta'\alpha'^{-1}$, there exists an unbalanced cycle C in $G[\![X]\!]$ (which is a subgraph of G) such that $\psi_G(\bar{C}) = \psi_G(C)$ by Proposition 11 (since $G[\![X]\!]$ is not balanced, and the label of any unbalanced cycle in $G[\![X]\!]$ is self-inversed by $(*)$), which contradicts Lemma 19.

Otherwise, i.e., if $|l(G[\![X]\!];x,y)| = 1$ or $l(G[\![X]\!];x,y) = \{\alpha',\beta'\}$ for some $\alpha',\beta' \in \Gamma$ with $\alpha'\beta'^{-1} \neq \beta'\alpha'^{-1}$, we can construct a smaller counterexample by the 2-contraction of X (by Definition 15 and Lemma 17-(3)), a contradiction. It should be noted that $(G[\![X]\!],x,y) \in \mathcal{D}_{\alpha',\beta'}$ if $l(G[\![X]\!];x,y) = \{\alpha',\beta'\}$, since G is a minimal counterexample and $G[\![X]\!]$ is a proper subgraph of G by the definition of the term "2-contractible" (see Definition 7). □

Claim 22. *There is no* 3*-contractible vertex set in* G*.*

Proof. Suppose to the contrary that G contains a 3-contractible vertex set $X \subseteq V \setminus \{s,t\}$. The minimality of G implies $(G/_3X,s,t) \in \mathcal{D}_{\alpha,\beta}$, which means that there exists a sequence $G_0, G_1, \ldots, G_r = G/_3X$ such that $G_0 = (\{s,t\},\{e_\alpha,e_\beta\})$, and G_{i-1} is obtained from G_i by some appropriate 2-contraction (cf. the proof of Lemma 17). We show that almost the same 2-contractions can be applied to G, which implies $(G,s,t) \in \mathcal{D}_{\alpha,\beta}$, a contradiction.

Let j be the maximum index such that $N_G(X) \cap (V(G_j) \setminus V(G_{j-1})) \neq \emptyset$. We then have $1 \leq j \leq r$ since $|V(G_0)| = 2$ and $|N_G(X)| = 3$, and we can apply to G the same 2-contractions as that to construct G_j from $G_r = G/_3X$. Let H_j be the resulting graph, $Y := V(G_j) \setminus V(G_{j-1})$ (i.e., $G_{j-1} = G_j/_2Y$), and $Z := X \cup Y \subseteq V(H_j)$. Then, $N_{H_j}(X) \subseteq Y \cup N_{G_j}(Y)$ since x and y are adjacent in G_j for any distinct $x,y \in N_{H_j}(X)$. Hence, X is 3-contractible also in $H_j[\![Z]\!]$, and we have $G_j[\![Y]\!] = H_j[\![Z]\!]/_3X - E(N_{G_j}(Y))$. This implies that the 2-contraction of Z in H_j does not violate the condition of $\mathcal{D}_{\alpha,\beta}$ (see Definition 15) since neither does that of Y in G_j, and $H_j/_2Z = G_j/_2Y$. Thus we have $(G,s,t) \in \mathcal{D}_{\alpha,\beta}$, a contradiction. □

Fix an arbitrary arc $e_0 = sv_0 \in \delta_G(s)$ leaving s, and let $G' := G - e_0$. Note that G contains no arc between s and t by Claim 21, and hence $v_0 \neq t$. We next show the following claims, which lead to $(G',s,t) \in \mathcal{D}_{\alpha,\beta}$.

Claim 23. $(G',s,t) \in \mathcal{D}$.

Proof. By Lemma 2, it suffices to show that $G' + r + rs + rt$ is 2-connected. Suppose to the contrary that it is not 2-connected, i.e., there exists a 1-cut $w \in V$ separating some vertex from both s and t (note that possibly $w \in \{s,t\}$). If $w = s$, then $G - s$ is not connected, which contradicts $(G,s,t) \in \mathcal{D}$. Otherwise, $\{s,w\}$ is a 2-cut in G, and hence G contains a 2-contractible vertex set $X \subseteq V \setminus \{s,t\}$ with $N_G(X) = \{s,w\}$, which contradicts Claim 21. □

Claim 24. $l(G';s,t) = \{\alpha,\beta\}$.

Proof. Since each s–t path in G' is also in G, $l(G';s,t) \subseteq l(G;s,t) = \{\alpha,\beta\}$. Suppose to the contrary that $|l(G';s,t)| = 1$. Then, G' is balanced by Lemma 5 and Claim 23, and hence $G - s$ is also balanced. This implies that $(G,s,t) \in \mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}_{\alpha,\beta}$ (in Case (A) in Definition 13), a contradiction. □

By Claims 23 and 24 and the minimality of G, we have $(G',s,t) \in \mathcal{D}_{\alpha,\beta}$. We consider the following two cases separately: when $(G',s,t) \in \mathcal{D}^1_{\alpha,\beta}$ and when not.

5.4 When $(G', s, t) \in \mathcal{D}^1_{\alpha,\beta}$ (Case 1).

By Claim 22, if G' contains a 3-contractible vertex set $X \subseteq V \setminus \{s,t\}$, then X must contain the head v_0 of e_0. Hence, if we choose a maximal 3-contractible vertex set X, then we have $(G'/_3X, s, t) \in \mathcal{D}^0_{\alpha,\beta}$. Let us define $\tilde{G} := G'/_3X$ in this case, and $\tilde{G} := G'$ otherwise (i.e., if G' contains no 3-contractible vertex set). Note that $(\tilde{G}, s, t) \in \mathcal{D}^0_{\alpha,\beta}$. We discuss the three cases in Definition 13 separately. Recall that we may assume that $\beta = 1_\Gamma$ and $\alpha^{-1} \neq \alpha$ (i.e., $\alpha^2 \neq 1_\Gamma$).

Case 1.1. When $(\tilde{G}, s, t)$ is in Case (A).

Note that any 3-contraction does not make an effect on this situation (i.e., either all unbalanced cycles in G' intersect s, or they do t) since it just replaces a balanced subgraph with a balanced triangle, and hence we may assume that $\tilde{G} = G'$ and G' satisfies the condition of Case (A) (by shifting at some vertices in $V \setminus \{s,t\}$ in advance of removing e_0 if necessary). Since G contains no 2-contractible vertex set, $G - \{s,t\}$ is connected, which implies that there exists a v_0–w path in $G - \{s,t\}$ for each neighbor $w \in N_G(t)$ (recall that $v_0 \neq t$). Therefore, if $e_0 = sv_0 \in \delta_G(s)$ violates the condition of Case (A) (i.e., $\psi_G(e_0) \notin \{1_\Gamma, \alpha\}$ in the former case, and $\psi_G(e_0) \neq 1_\Gamma$ in the latter case), then it is easy to see that $|l(G; s,t)| \geq 3$ (see Figs. 9 and 10). Note that we use Lemma 20 in the latter case (let $P_1 := (s)$ and $P_2 := (s, e_0, v_0)$).

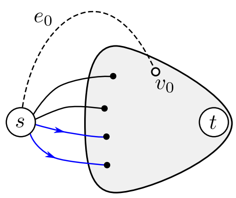

Figure 9: The former of Case (A).

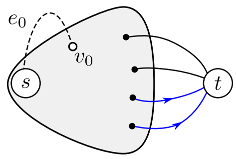

Figure 10: The latter of Case (A).

Case 1.2. When $(\tilde{G}, s, t)$ is in Case (B).

If $\tilde{G} = G'$, then it is easy to see $|l(G; s,t)| \geq 3$ by Lemma 20, since G contains no parallel arc with the same label (see Fig. 11). Otherwise, $\tilde{G} = G'/_3X$ for some $X \subseteq V \setminus \{s,t\}$. If $N_{G'}(X) = \{s, v_1, v_2\}$, then $G[\![X]\!]$ is not balanced by Claim 22, and hence $|l(G; s,t)| \geq 3$ by Lemma 20 (e.g., we can take two s–v_1 paths P_1 and P_2 in $G[\![X]\!]$ with $\psi_G(P_1) \neq \psi_G(P_2)$).

Suppose that $N_{G'}(X) = \{v_3, v_4, t\}$ (see Fig. 12). If there exist two disjoint paths between $\{v_0, t\}$ and $\{v_3, v_4\}$ in $G[\![X]\!]$, then $|l(G; s,t)| \geq 3$ by Lemma 20 (e.g., we can take two s–v_1 paths P_1 and P_2 in $G[\![X + v_3]\!]$ with $\psi_G(P_1) \neq \psi_G(P_2)$ and $l(G - (V(P_i) - v_1); v_1, t) = \{1, \alpha\}$ $(i = 1, 2)$, if $G[\![X]\!]$ contains disjoint v_0–v_3 path and t–v_4 path). Otherwise, by Menger's theorem, $G[\![X]\!]$ contains a 1-cut $w \in X$ separating $\{v_0, t\}$ from $\{v_3, v_4\}$ (possibly $w = v_0$). In this case, $\{s, w\}$ is a 2-cut in G, which contradicts Claim 21.

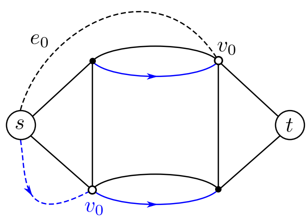

Figure 11: Case (B) ($\tilde{G} = G'$).

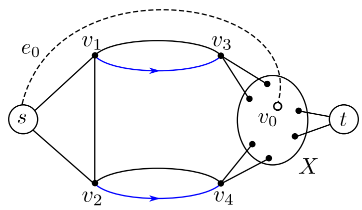

Figure 12: Case (B) ($\tilde{G} = G'/_3X$).

Case 1.3. When $(\tilde{G}, s, t)$ is in Case (C).

Suppose that $\tilde{G} = (\tilde{V}, \tilde{E})$ is embedded with the conditions in Lemma 18 (we apply shifting at each vertex $v \in V \setminus \{s, t\}$ to G in advance of the construction of $\tilde{G}$ if necessary). Let $\tilde{E}^i \subseteq \tilde{E}$ be the arc set corresponding to $E^i \subseteq E$ in Lemma 18 for each $i = 0, 1$, and we refer to the path $P = (s = u_0, e_1, u_1, \ldots, e_l, u_l = t)$ along the outer boundary of $\tilde{G} - \tilde{E}^1$ as P itself.

In what follows, we derive a contradiction by showing that $(G, s, t) \in \mathcal{D}_{1_\Gamma, \alpha}$, $\gamma \in l(G; s, t)$ for some $\gamma \in \Gamma \setminus \{1_\Gamma, \alpha\}$ (in particular, $\gamma = \alpha^2$ or α^{-1}), or G contains a contractible vertex set (which contradicts Claims 21 or 22). Note that $(\tilde{G}, s, t) \in \mathcal{D}$ follows from $(G', s, t) \in \mathcal{D}$, and hence $\tilde{G} - s$ is connected. Since every arc in $\tilde{E}^1$ connects two vertices on the path P in $\tilde{G} - \tilde{E}^1$, $\tilde{G} - \tilde{E}^1 - s$ is also connected. Hence, we have $\psi_G(e_0) \in l(G; s, t) = \{1_\Gamma, \alpha\}$, and consider the following two cases separately: when $\psi_G(e_0) = 1_\Gamma$, and when $\psi_G(e_0) = \alpha$.

Note that we have $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$. To see this, suppose that $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) = \emptyset$. In this case, $G - s$ as well as $\tilde{G} - s$ is balanced, which implies that $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta} \subseteq \mathcal{D}_{\alpha,\beta}$ in Case (A) in Definition 13. We can also see $\tilde{E}^1 \setminus \delta_{\tilde{G}}(t) \neq \emptyset$ in the same way.

We first discuss the case when $\tilde{G} = G'$, and later explain that the case when $\tilde{G} = G'/_3 X$ for some $X \subseteq V \setminus \{s, t\}$ can be dealt with in almost the same way with the aid of Theorem 1. Assume $\tilde{G} = G' = G - e_0$, and let $\tilde{F}_0$ and $\tilde{F}'_0$ denote the outer faces of $\tilde{G}$ and $\tilde{G} - s$, respectively.

Case 1.3.1. When $\psi_G(e_0) = 1_\Gamma$.

Let us begin with an easy case: when $v_0 \in V(\mathrm{bd}(\tilde{F}'_0))$.

Case 1.3.1.1. Suppose that $v_0 \in V(\mathrm{bd}(\tilde{F}'_0)) \setminus V(P)$. In this case, we can embed $G = \tilde{G} + e_0$ on a plane by adding $e_0 = sv_0$ on $\tilde{F}_0$ so that (G, s, t) satisfies the conditions of Case (C), a contradiction.

Case 1.3.1.2 (Fig. 13)**.** Otherwise, $v_0 = u_h \in V(\mathrm{bd}(\tilde{F}'_0)) \cap V(P)$. Take an s–t path P' so that $(P' \cup P) - s$ forms the outer boundary of $\tilde{G} - \tilde{E}^1 - s$. Let j be the minimum index such that $E(P[u_j, t]) \subseteq E(P')$, and i the index such that $P[u_i, u_j] \cup P'[u_i, u_j]$ forms a cycle (i.e., they intersect only at u_i and u_j).

Take an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that $j' - i'$ is maximized. If $j' \leq i$, then G contains a 2-cut $\{s, u_i\}$ separating $u_{i-1} \neq s$ from $t \neq u_i$, which contradicts Claim 21. Hence, we have $i < j'$.

If $v_0 = u_h \in V(P) \cap V(P')$ or $h \leq i'$ ((a) in Fig. 13), then we can embed $e_0 = sv_0$ without violating the conditions of Case (C). Otherwise, we have $j' \leq h < j$ ((b) in Fig. 13) since $u_h = v_0 \in V(\mathrm{bd}(\tilde{F}'_0)) \cap V(P)$. In this case, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, $\bar{P}[u_h, u_{j'}]$, $\bar{e}'$, $P[u_{i'}, u_i]$, $P'[u_i, u_j]$, and $P[u_j, t]$ if $0 < i' \leq i$.

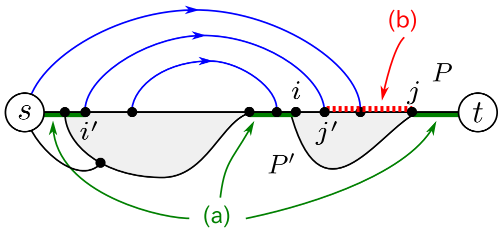

Figure 13: Case 1.3.1.2.

Otherwise, $v_0 \notin V(\mathrm{bd}(\tilde{F}'_0))$. Take a path Q in $\tilde{G} - \tilde{E}^1 - E(P) - s$ from $u_i \in V(P)$ to $u_j \in V(P)$ with $0 < i < j$ so that $Q \cup P[u_i, u_j]$ forms a cycle that encloses v_0 (possibly $v_0 \in V(P)$), i.e., $V(Q \cup P[u_i, u_j])$ separates v_0 from both of s and t in $\tilde{G}$ (or $v_0 = u_h \in V(P)$

with $i < h < j$). If there are multiple choices of Q, then choose Q so that the region enclosed by $Q \cup P[u_i, u_j]$ is maximized.

If $V(Q)$ separates v_0 from $V(P)$ in $\tilde{G}$, then G contains a 3-contractible vertex set $X \subseteq V \setminus V(P)$ such that $v_0 \in X$ and $N_G(X) = \{s, w_1, w_2\}$, which contradicts Claim 22, where $w_1, w_2 \in V(Q)$ are the vertices closest $u_i, u_j \in V(P) \cap V(Q)$, respectively, among those which are reachable from v_0 in $\tilde{G}$ without intersecting Q in between. Thus we can take a v_0–u_h path R in $\tilde{G} - V(Q)$ (possibly of length 0, i.e., $v_0 = u_h$) with $i < h < j$. If there are multiple choices of R, then choose R so that h is maximized under the condition that $V(R) \cap V(P) = \{u_h\}$.

Case 1.3.1.3 (Fig. 14)**.** Suppose that there is no arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ incident to an inner vertex on $P[u_i, u_j]$. If every arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ enters a vertex on $P[s, u_i] \cup P[u_j, t]$, then G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_i, u_j\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_i, u_j\}$, a contradiction. Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ enters a vertex on $P[s, u_i]$. Then, G contains a 2-cut $\{s, u_i\}$ separating u_{i-1} from t (note that $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ implies that $i > 1$), which contradicts Claim 21.

Case 1.3.1.4 (Fig. 15)**.** Suppose that there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ such that $i' < h$ and $i < j' < j$. In this case, we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $P[u_h, u_{j'}]$, $\bar{e}'$, $\bar{P}[u_{i'}, u_i]$, Q, and $P[u_j, t]$ if $i \leq i'$ and $h \leq j'$.

Case 1.3.1.5 (Fig. 16)**.** Suppose that every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ connects two vertices on $P[u_h, t]$. In this case, every arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ also enters a vertex on $P[u_h, t]$, and $v_0 \neq u_h$ since $v_0 \notin V(\mathrm{bd}(\tilde{F}_0'))$. Let w be the vertex closest to u_j among those on Q which are reachable from v_0 in $G - u_h$ without intersecting Q in between. By the maximality of j and h (i.e., the choice of Q and R), $\{s, u_h, w\}$ separates $v_0 \in V \setminus \{s, u_h, w\}$ from $V(P[u_h, t])$ in G, and hence G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_h, w\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_h, w\}$, a contradiction.

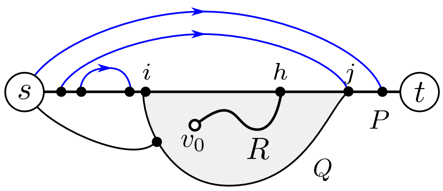

Figure 14: Case 1.3.1.3.

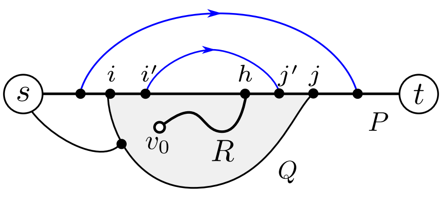

Figure 15: Case 1.3.1.4.

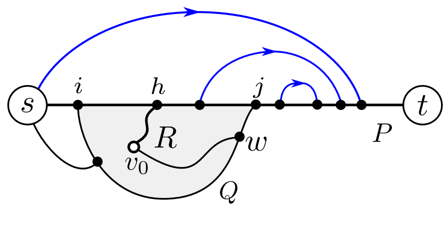

Figure 16: Case 1.3.1.5.

These three cases imply that there exists an arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ entering a vertex on $P[u_j, t]$. To see this, suppose to the contrary that every such arc enters a vertex on $P[u_1, u_{j-1}]$, and take $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ so that $j' - i'$ is maximized. We may assume $i < j'$ by Case 1.3.1.3, and hence $h \leq i'$ by Case 1.3.1.4, which leads to the condition of Case 1.3.1.5, a contradiction. This implies also that no arc in $\tilde{E}^1 \cap \delta_{\tilde{G}}(s)$ enters a vertex on $P[u_1, u_{j-1}]$.

Case 1.3.1.6 (Fig. 17)**.** Suppose that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ leave the same vertex $u_{i^*} \in V(P)$ with $i^* < h$. In this case, by Case 1.3.1.4, we may assume that every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enters a vertex on $P[u_j, t]$. Then, since $\{s, u_{i^*}, u_j\}$ separates $v_0 \in V \setminus \{s, u_{i^*}, u_j\}$ from $V(P[u_j, t])$ in G, there exists a 3-contractible vertex set $X \subseteq V \setminus \{s, u_{i^*}, u_j\}$ in G such that $v_0 \in X$ and $N_G(X) = \{s, u_{i^*}, u_j\}$, a contradiction.

Case 1.3.1.7 (Fig. 18)**.** Suppose that all arcs in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ enter the same vertex $u_{j^*} \in V(P)$ with $j \le j^*$. In this case, $\{s, u_j, u_{j^*}\}$ separates $v_0 \in V \setminus \{s, u_j, u_{j^*}\}$ from $V(P[u_j, t])$ in G, and hence G contains a contractible vertex set $X \subseteq V \setminus \{s, u_j, u_{j^*}\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_j, u_{j^*}\}$, a contradiction. Note that $u_{j^*} \neq t$ by $\tilde{E}^1 \setminus \delta_{\tilde{G}}(t) \neq \emptyset$.

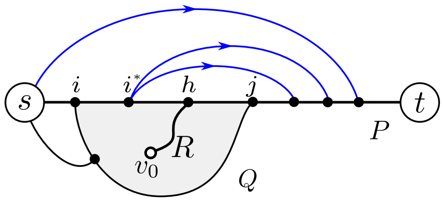

Figure 17: Case 1.3.1.6.

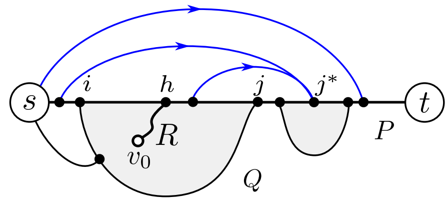

Figure 18: Case 1.3.1.7.

Otherwise, there exist two arcs $e_1 = u_{i_1}u_{j_1}$ and $e_2 = u_{i_2}u_{j_2}$ in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ such that $i_2 < i_1 < j_1 < j_2$ by Cases 1.3.1.6 and 1.3.1.7. We choose e_2 so that $j_2 - i_2$ is maximized. We then have $i_2 < h$ by Case 1.3.1.5, and $j \le j_2$ by the argument just after Case 1.3.1.5. Since there exists an arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ incident to an inner vertex on $P[u_i, u_j]$ by Case 1.3.1.3, we can choose e_1 so that $i < i_1$ (which is obvious if $i \le i_2$, and follows from Case 1.3.1.4 otherwise). We then have $h < j_1$, since otherwise we have $i < i_1 < j_1 \le h < j$, which implies that e_1 satisfies the condition of Case 1.3.1.4. We choose e_1 so that i_1 is minimized under the condition that $i < i_1$.

Case 1.3.1.8 (Fig. 19)**.** Suppose that $j \le i_1$. In this case, $\{s, u_{i_2}, u_j\}$ separates $v_0 \in V \setminus \{s, u_{i_2}, u_j\}$ from $P[u_j, t]$ in G, and hence G contains a 3-contractible vertex set $X \subseteq V \setminus \{s, u_{i_2}, u_j\}$ such that $v_0 \in X$ and $N_G(X) = \{s, u_{i_2}, u_j\}$, a contradiction.

Case 1.3.1.9 (Figs. 21 and 22)**.** Suppose that $j_2 = j$. We then have $h \le i_1$ by $i < i_1 < j_1 < j_2 = j$ and Case 1.3.1.4. Let h^* be the maximum index such that there exists a w–u_{h^*} path R^* in $\tilde{G} - u_j$ for some $w \in (V(Q) \setminus V(P)) + v_0$ such that $V(R^*) \cap V(Q) \subseteq \{w\}$ and $V(R^*) \cap V(P) = \{u_{h^*}\}$. Note that $h \le h^*$. If $i_1 < h^*$, then we have $h < h^*$ because of $h \le i_1$. In this case (see Fig. 21), since R and R^* are disjoint by the maximality of h and h^*, we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $P[u_h, u_{i_1}]$, e_1, $\bar{P}[u_{j_1}, u_{h^*}]$, $\bar{R}^*$, $\bar{Q}[w, u_i]$, $\bar{P}[u_i, u_{i_2}]$, e_2, and $P[u_j, t]$ if $h^* \le j_1$ and $i_2 \le i$. Otherwise (i.e., if $h^* \le i_1$), by the minimality of i_1 and the maximality of h^*, there exists a 2-cut $\{u_{h^*}, u_j\}$ separating u_{j_1} from u_i ($i < h \le h^* \le i_1 < j_1 < j_2 = j$) in G (see Fig. 22), a contradiction.

Case 1.3.1.10 (Fig. 20)**.** Otherwise, we have $i < i_1 < j < j_2$ (also recall that $i_2 < i_1 < j_1 < j_2$ and $i_2 < h < j_1$). In this case, we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, e.g., by concatenating e_0, R, $\bar{P}[u_h, u_{i_1}]$, e_1, $\bar{P}[u_{j_1}, u_j]$, $\bar{Q}$, $P[u_i, u_{i_2}]$, e_2, and $P[u_{j_2}, t]$ if $i_1 \le h$, $j \le j_1$, and $i \le i_2$.

Case 1.3.2. When $\psi_G(e_0) = \alpha$.

This case is rather easier than Case 1.3.1. Note that, if there exists a v_0–t path of label α in $\tilde{G} = G' = G - e_0$, then we can construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G, a contradiction, by extending the v_0–t path using $e_0 = sv_0$. Hence, we may assume that $\tilde{G}$ contains no such path.

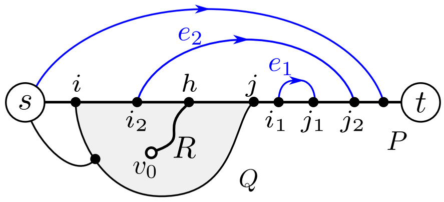

Figure 19: Case 1.3.1.8.

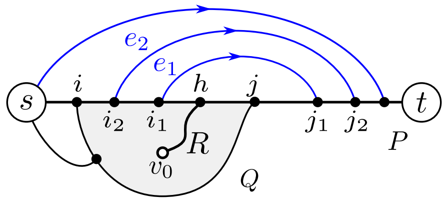

Figure 20: Case 1.3.1.10.

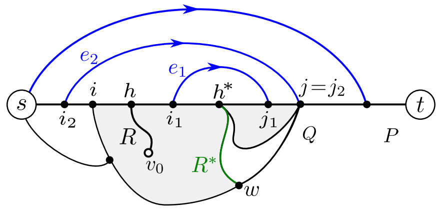

Figure 21: Case 1.3.1.9 (label α^2).

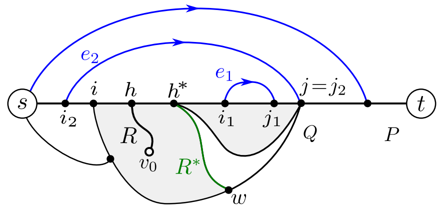

Figure 22: Case 1.3.1.9 (a 2-cut $\{u_{h^*}, u_j\}$).

Case 1.3.2.1. Suppose that $v_0 = u_h \in V(P)$. If there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $h < j'$, then we can construct a v_0–t path of label α, a contradiction, e.g., by concatenating $P[u_h, u_{i'}]$, e', and $P[u_{j'}, t]$ if $h \leq i'$. Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ connects two vertices on $P[u_1, u_h]$. Hence, we can embed $e_0 = su_h$ without violating the conditions of Case (C) in Definition 13 (cf. Lemma 18).

Case 1.3.2.2 (Fig. 23)**.** Otherwise, $v_0 \notin V(P)$. Let i and j be the minimum and maximum indices, respectively, such that there exist a v_0–u_i path Q and a v_0–u_j path R in $\tilde{G} - \tilde{E}^1 - s$ that do not intersect P in between. If there exists an arc $e' = u_{i'}u_{j'} \in \tilde{E}^1 \setminus \delta_{\tilde{G}}(s)$ with $i < j'$, then we can construct a v_0–t path of label α, a contradiction, e.g., by concatenating Q, $\bar{P}[u_i, u_{i'}]$, e', and $P[u_{j'}, t]$ if $i' \leq i$.

Otherwise, every arc in $\tilde{E}^1 \setminus \delta_{\tilde{G}}(s) \neq \emptyset$ connects two vertices on $P[u_1, u_i]$. Since G contains no 3-contractible vertex set (by Claim 22), there exists an arc from s to the connected component of $\tilde{G} - \{s, u_i, u_j\}$ that contains v_0 with label 1_Γ in $\tilde{G}$. Hence, because of the minimality of i and the planarity of $\tilde{G}$, there is no path from an inner vertex on $P[s, u_i]$ to a vertex on $P[u_j, t]$ in $\tilde{G} - \tilde{E}^1 - s$ which does not intersect P in between. This implies that G contains a 2-cut $\{s, u_i\}$ separating $u_1 \neq u_i$ from t, a contradiction.

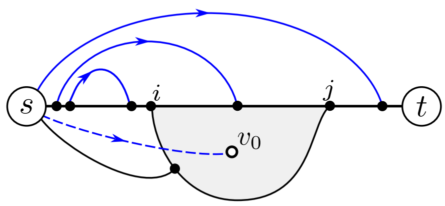

Figure 23: Case 1.3.2.2.

Case 1.3.3. When $\tilde{G} = G'/_3X$ for some $X \subseteq V \setminus \{s, t\}$.

Recall that X must contain v_0 by Claim 22. Suppose that $N_{G'}(X) = \{y_1, y_2, y_3\}$. Since $\tilde{G}$ is embedded as Lemma 18, the resulting triangle $y_1y_2y_3$ of the 3-contraction of X (which is a balanced cycle by the definition) consists of either three arcs in $\tilde{E}^0$ or one arc in $\tilde{E}^0$ and two arcs in $\tilde{E}^1$. Without loss of generality (by the symmetry of y_1, y_2, y_3), assume that the arc between y_2 and y_3 is in $\tilde{E}^0$, i.e., $l(G'[\![X]\!]; y_2, y_3) = 1_\Gamma$. Then, by shifting at vertices in X in advance of removing $e_0 = sv_0$ from G if necessary, we may assume that the label of every arc in $G'[\![X]\!] - y_1$ is 1_Γ and in $\delta_{G'[\![X]\!]}(y_1)$ is γ, where γ is a fixed element in $\{1_\Gamma, \alpha, \alpha^{-1}\}$ and all arcs in $\delta_{G'[\![X]\!]}(y_1)$ are assumed to enter y_1 (recall that $G'[\![X]\!]$ is balanced by Definition 8).

Let $\tilde{G}'$ be the Γ-labeled graph obtained from G' by the following procedure:

- merge all vertices in X into v_0,
- identify parallel arcs with the same label as a single arc, and
- for each $\{i, j, k\} = \{1, 2, 3\}$, add an arc from y_j to y_k with label $l(G'[\![X]\!]; y_j, y_k)$ if there is no such arc and there are disjoint v_0–y_i path and y_j–y_k path in $G'[\![X]\!]$ (note that otherwise, by Theorem 1, $G'[\![X]\!]$ can be embedded on a plane so that v_0, y_j, y_i, y_k are on the outer boundary in this order).

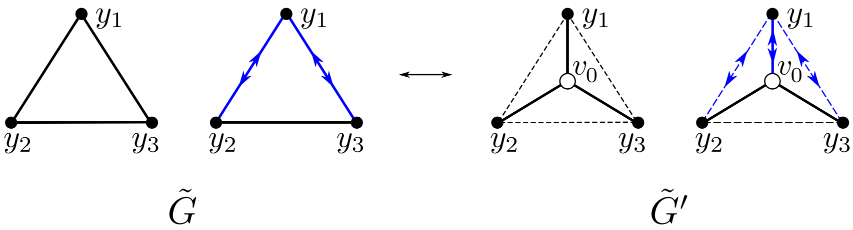

Figure 24: Corresponding parts of $\tilde{G}$ and $\tilde{G}'$.

Since $\tilde{G}$ is embedded as Lemma 18, we can naturally embed $\tilde{G}'$ so (see Fig. 24). By the same argument for $\tilde{G}'$ as Cases 1.3.1 and 1.3.2, we can derive a contradiction in this case. Note that, if we can construct an s–t path of label $\gamma \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in $\tilde{G}' + e_0$, then it can be expanded to one in $G = G' + e_0$ (which may use disjoint v_0–y_i path and y_j–y_k path in $G'[\![X]\!]$). Besides, if we can embed $\tilde{G}' + e_0$ as Lemma 18, then the embedding can be expanded to one of G without violating the conditions, since any embedding of $\tilde{G}'$ with v_0 exposed on the outer boundary can be expanded to one of G' so by Theorem 1. Note that, for any k-cut ($k \in \{2, 3\}$) separating some vertex set from $\{v_0, y_1, y_2, y_3\}$ in $G'[\![X]\!]$, we can perform the k-contraction, respectively, which emulates the operation in Condition 3 in Theorem 1, since $G'[\![X]\!]$ is balanced.

5.5 When $(G', s, t) \in \mathcal{D}_{\alpha,\beta} \setminus \mathcal{D}^1_{\alpha,\beta}$ (Case 2).

In this case, G' contains a 2-contractible vertex set $X \subseteq V \setminus \{s, t\}$ by the definition of $\mathcal{D}_{\alpha,\beta}$ (see Definition 15). Due to the previous section, we may assume that this situation occurs regardless of the choice of the arc $e_0 = sv_0 \in \delta_G(s)$, which has at least two possibilities by Lemma 17-(2). We first show a useful claim about such a vertex set (in fact, slightly more general).

Claim 25. *Let $X \subseteq V \setminus \{s, t\}$ be a vertex set with $N_G(X) = \{s, x, y\}$ for some distinct $x, y \in V$ (see Fig. 25). Then, $s \notin \{x, y\}$, $G[\![X]\!]$ is not balanced, and $(G[\![X]\!] - x, s, y) \in \mathcal{D}$. Moreover, if $|l(G[\![X]\!]; s, y)| = 1$, then $X = \{v\}$ for some $v \in V \setminus \{s, x, y\}$ and $G[\![X]\!]$ consists of a single arc between s and v, one between v and y, and two parallel arcs between v and x (see Fig. 26).*

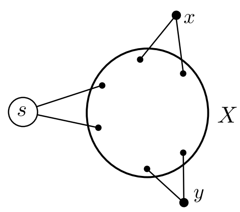

Figure 25: The situation of Claim 25.

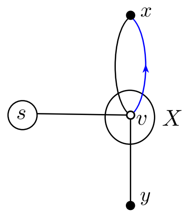

Figure 26: When $|l(G[\![X]\!]; s, y)| = 1$.

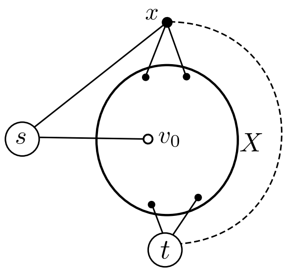

Figure 27: Case 2.1.1.

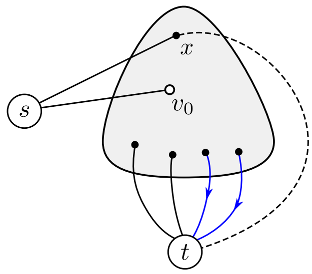

Figure 28: Case 2.1.1.1.

Proof. If $s \in \{x, y\}$, then X is 2-contractible in G, which contradicts Claim 21. Besides, if $G[\![X]\!]$ is balanced, then X is 3-contractible in G, which contradicts Claim 22.

Suppose to the contrary that $(G[\![X]\!] - x, s, y) \notin \mathcal{D}$. Then, $G[\![X]\!] - x + sy$ contains a 1-cut $w \in X \cup \{s, y\}$ by Lemma 2. The vertex set of the connected component of $G[\![X]\!] - \{w, x\} + sy$ that contains none of s and y is separated from both of s and t by $\{w, x\}$ in G (possibly $t \in \{w, x\}$), and hence it is 2-contractible, a contradiction.

Moreover, suppose that $|l(G[\![X]\!]; s, y)| = 1$, which leads to $(G[\![X]\!], s, y) \notin \mathcal{D}$ by Lemma 5. Then, $G[\![X]\!] - x$ is balanced since $(G[\![X]\!] - x, s, y) \in \mathcal{D}$, which also implies that $G[\![X]\!] + sy$ contains a unique 1-cut $w \in X$. The 1-cut w separates x from the balanced component $G[\![X]\!] - x$, and hence there are two parallel arcs between w and x (which form an unbalanced cycle). Besides, if $X - w \neq \emptyset$, then G contains a contractible vertex set $Y \subseteq X - w$ with $N_G(Y) \subseteq \{s, w, y\}$, a contradiction. Thus we have done. $\square$

Choose a minimal 2-contractible vertex set X in G', and let $N_{G'}(X) = \{x, y\}$. We then have $v_0 \in X$ and $s \notin \{x, y\}$ by Claim 21 ($G = G' + e_0$ contains no 2-contractible vertex set), and $(G'[\![X]\!], x, y) \in \mathcal{D}^1_{\alpha', \beta'}$ for some distinct $\alpha', \beta' \in \Gamma$ with $\alpha' \beta'^{-1} \neq \beta' \alpha'^{-1}$ by Lemma 19 and Claims 23–25. Besides, $G[X]$ must be connected, since otherwise some connected component of $G[X]$ does not contain v_0 and hence its vertex set is contractible in G, which contradicts Claim 21 or 22.

<u>**Case 2.1.** When $t \in \{x, y\}$.</u>

Without loss of generality (by the symmetry of x and y), we may assume that $y = t$.

<u>**Case 2.1.1.** When $V = X \cup \{s, x, t\}$.</u>

Recall that G contains no arc between s and t. Hence, by Lemma 17-(2), G contains an arc between s and x, and there exists exactly one such arc $e = sx \in E$ (see Fig. 27), since $(G'[\![X]\!], x, y) \in \mathcal{D}^1_{\alpha', \beta'}$ and $|l(G; s, t)| = 2$. We assume $\psi_G(e) = 1_\Gamma$ by shifting at x if necessary. In the same way as the previous section, let $\tilde{G} := G'[\![X]\!] /_3 Y$ for a maximal 3-contractible vertex set $Y \subseteq X$ with $v_0 \in Y$ if exists, and $\tilde{G} := G'[\![X]\!]$ otherwise. We then have $(\tilde{G}, x, t) \in \mathcal{D}^0_{\alpha, \beta}$, and consider the three cases in Definition 13 separately.

Case 2.1.1.1. Suppose that $(\tilde{G}, x, t)$ is in the latter case of Case (A) (see Fig. 28). We may assume that the label of every arc in $E(X + x)$ is 1_Γ (by shifting at vertices in X if necessary). If $\psi_G(e_0) = 1_\Gamma$, then obviously $(G, s, t) \in \mathcal{D}^0_{\alpha, \beta}$. Otherwise (i.e., if $\psi_G(e_0) \neq 1_\Gamma$), since $G[X]$ is connected, there exists a v_0–w path in $G'[\![X]\!]$ for each neighbor $w \in N_G(t)$, and hence $|l(G, s, t)| \geq 3$ by Lemma 20. Note that any 3-contraction does not make an effect on the above argument.

Case 2.1.1.2. Suppose that $(\tilde{G}, x, t)$ is in the former case of Case (A) (see Fig. 29). We may assume that the label of every arc in $E(X + t)$ is 1_Γ and in $\delta_G(x) - e$ leaving x is 1_Γ or α with $\alpha^{-1} \neq \alpha$ (recall that we may assume $\beta = 1_\Gamma$ by Lemma 17-(1)). Note again that any 3-contraction does not make an effect on whether $(\tilde{G}, x, t)$ is in Case (A) or not, and hence we may assume that $\tilde{G} = G'[\![X]\!]$.

Let H be the graph obtained from $G - s$ (which coincides with $G'[\![X]\!]$ if $xt \notin E$) by splitting x into two vertices x_0 and x_1 so that every arc leaving x in $G - s$ with label $\alpha^i \in \{1_\Gamma, \alpha\}$ leaves x_i in H for each $i = 0, 1$ (see Fig. 30).

Since $l(G; s, t) = \{1_\Gamma, \alpha\}$, either $\psi_G(e_0) = 1_\Gamma$ or $\psi_G(e_0) = \alpha$. Suppose that $\psi_G(e_0) = 1_\Gamma$. If H contains disjoint v_0–x_1 path P and x_0–t path Q, then we can construct an s–t path of label $\alpha^{-1} \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G by concatenating e_0, P, and Q with identifying $x_0, x_1 \in V(H)$ as $x \in V$. Otherwise, by Theorem 1, H can be embedded on a plane so that $v_0, x_0, x_1, t \in V(H)$ are on the outer boundary in this order (note that if there exists a vertex set $Y \subseteq V(H) \setminus \{v_0, x_0, x_1, t\} = V \setminus \{v_0, x, t\}$ such that $|N_H(Y)| \leq 3$, then either $|N_G(Y)| \leq 2$ or $|N_G(Y)| \leq 3$ and $G[\![Y]\!]$ is balanced, which contradicts Claim 21 or 22, respectively). This embedding can be easily extended to an embedding of G by merging $x_0, x_1 \in V(H)$ into $x \in V$ and by adding s, $e_0 = sv_0$, and $e = sx$, and the resulting embedding satisfies the conditions of Case (C) in Definition 13 (cf. Lemma 18), which implies $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$, a contradiction.

Otherwise, $\psi_G(e_0) = \alpha$. Also in this case, by a similar argument to the above, we can either construct an s–t path of label $\alpha^2 \in \Gamma \setminus \{1_\Gamma, \alpha\}$ in G by concatenating e_0 and disjoint v_0–x_0 path P and x_1–t path Q with identifying $x_0, x_1 \in V(H)$ as $x \in V$, or embed G so that (G, s, t) is in Case (C).

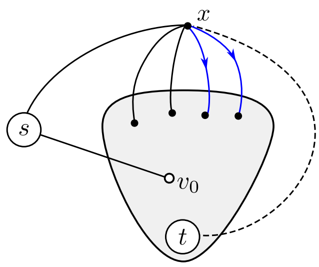

Figure 29: Case 2.1.1.2.

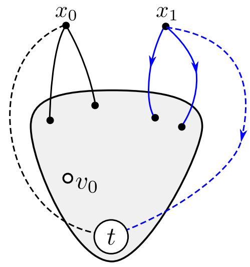

Figure 30: H in Case 2.1.1.2.

Case 2.1.1.3. Suppose that $(\tilde{G}, x, t)$ is in Case (B). If $\tilde{G} = G'[\![X]\!]$, it is easy to confirm that $\{x\}$ is 3-contractible in G (if there is no arc between x and t) or $|l(G; s, t)| \geq 3$ (otherwise, i.e., if $xt \in E$) by Lemma 20 (see Fig. 31).

Otherwise (i.e., if $\tilde{G} = G'[\![X]\!]/_3 Y$ for some $Y \subseteq X$), we have either $N_{G'}(Y) = \{x, v_1, v_2\}$ or $N_{G'}(Y) = \{v_3, v_4, t\}$. Suppose that $N_{G'}(Y) = \{v_3, v_4, t\}$. In this case, we can derive a contradiction by Menger's Theorem in a similar way to Case 1.2. That is, $G'[\![Y]\!]$ contains either two disjoint paths between $\{v_0, t\}$ and $\{v_3, v_4\}$ or a 1-cut $w \in Y$ separating them (possibly $w = v_0$). In the former case, $|l(G; s, t)| \geq 3$ by Lemma 20, and in the latter case, G contains a 2-cut $\{x, w\}$ separating $\{v_3, v_4\}$ from $\{s, v_0, t\}$, which contradicts Claim 21.

Otherwise, $N_{G'}(Y) = \{x, v_1, v_2\}$ (see Fig. 32). If $xt \in E$, then we can similarly derive a contradiction by Menger's Theorem, i.e., either $|l(G; s, t)| \geq 3$ by Lemma 20 ($G'[\![Y]\!]$ contains two disjoint paths between $\{v_0, x\}$ and $\{v_1, v_2\}$) or G contains a 2-cut $\{w, t\}$ ($G'[\![Y]\!]$ contains a 1-cut $w \in Y + x$ separating $\{v_0, x\}$ and $\{v_1, v_2\}$). Otherwise, $N_G(Y + x) = \{s, v_1, v_2\}$. Since $Y + x$ is not 3-contractible in G by Claim 22, $G[\![Y + x]\!]$ is not balanced. If $|l(G[\![Y + x]\!]; s, v_1)| = 1$, then G contains a 3-contractible vertex set $Z \subseteq Y + x$ with $N_G(Z) = \{s, v_1, w\}$ for some $w \in Y$ (note that $G'[Y] = G[Y]$ is connected by Definition 8), a contradiction. Otherwise, i.e., if $|l(G[\![Y + x]\!]; s, v_1)| \geq 2$, we have $|l(G; s, t)| \geq 3$ by Lemma 20.

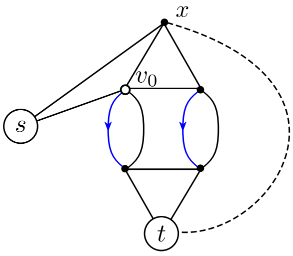

Figure 31: Case 2.1.1.3 ($\tilde{G} = G'[\![X]\!]$).

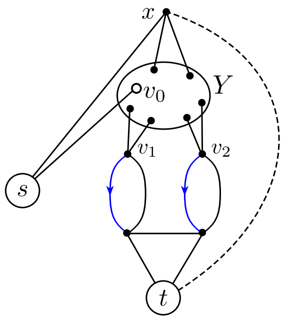

Figure 32: Case 2.1.1.3 ($\tilde{G} = G'[\![X]\!]/_3 Y$).

Case 2.1.1.4. Suppose that $(\tilde{G}, x, t)$ is in Case (C). In this case, by extending the x–t path P (in Lemma 18) to an s–t path using the arc $e = sx$, we can see that (G', s, t) (or $(G'/_3 Y, s, t)$ if $\tilde{G} = G'[\![X]\!]/_3 Y$) is also in Case (C) (see Fig. 33), which contradicts $(G', s, t) \notin \mathcal{D}^1_{\alpha,\beta}$.

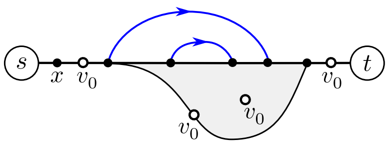

Figure 33: Case 2.1.1.4.

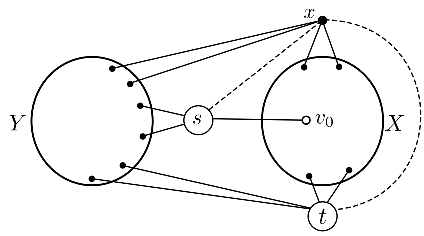

Figure 34: Case 2.1.2.

Case 2.1.2. When $V \setminus (X \cup \{s, x, t\}) \neq \emptyset$.

Let $Y := V \setminus (X \cup \{s, x, t\})$. Since Y is not 3-contractible in G by Claim 22, $G[\![Y]\!]$ is not balanced. We focus on the graph $G - X - t$, which coincides with $G[\![Y]\!] - t$ if $sx \notin E$ (see Fig. 34). We have $(G[\![Y]\!] - t, s, x) \in \mathcal{D}$ by Claim 25, and hence $(G - X - t, s, x) \in \mathcal{D}$. Suppose that $G - X - t$ is not balanced. In this case, $|l(G - X - t; s, x)| \geq 2$ by Lemma 5, and hence $|l(G; s, t)| \geq 3$ by Lemma 20 (recall that $(G'[\![X]\!], x, t) \in \mathcal{D}_{\alpha',\beta'}$).

Otherwise, $G - X - t$ is balanced. By Claim 25 (with the symmetry of s and t), we have $(G - X - s, x, t) \in \mathcal{D}$ and hence $|l(G - X - s, x, t)| \geq 2$. This implies that $G[\![X]\!] - t$ is balanced, since otherwise $|l(G; s, t)| \geq 3$ by Lemma 20 (note that $(G[\![X]\!] - t, s, x) \in \mathcal{D}$ by Claim 25). In this case, by Lemma 4, we may assume that $\psi_G(e) = 1_\Gamma$ for every $e \in E \setminus (\delta_G(t) + e_0)$ (by shifting at each $v \in V \setminus \{s, t\}$ if necessary).

If $\psi_G(e_0) = 1_\Gamma$, then $G - t$ is also balanced, and hence $(G, s, t) \in \mathcal{D}^0_{\alpha,\beta}$ in the latter case of Case (A) in Definition 13, a contradiction. Otherwise, we have $|l(G; s, t)| \geq 3$ by Lemma 20 (we choose $P_1 := (s, e_0, v_0)$ and P_2 as an arbitrary s–x path in $G - X - t$, there are two arcs entering t from X with distinct labels since X is not 3-contractible in G by Claim 22, and recall that $G[X]$ is connected as discussed just before starting Case 2.1), a contradiction.

Case 2.2. When $t \notin \{x, y\}$.

Suppose that $V = X \cup \{s, x, y, t\}$. Then, by the symmetry of x and y, we may assume that there exists an arc $e = sx \in \delta_G(s)$ such that $(G - e, s, t) \in \mathcal{D}_{\alpha,\beta} \setminus \mathcal{D}^1_{\alpha,\beta}$ (recall the discussion in the first paragraph of this section, Section 5.5). Besides, t is adjacent to both of x and y since otherwise $\{s, y\}$ or $\{s, x\}$ is a 2-cut in G, which contradicts Claim 21. Hence, by choosing e instead of e_0, we can reduce this case to Case 2.1 (since x and t are adjacent, t must be a neighbor of any 2-contractible vertex set in $G - e$ that contains x).

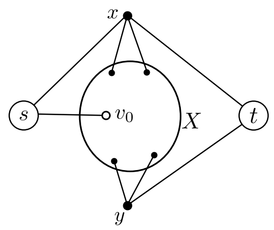

Figure 35: When $V = X \cup \{s, x, y, t\}$.

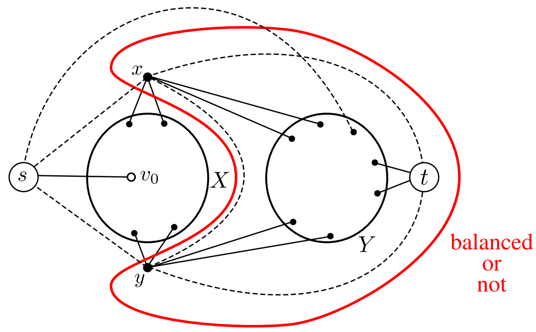

Figure 36: When $V \setminus (X \cup \{s, x, y, t\}) \neq \emptyset$.

In what follows, we assume that $Y := V \setminus (X \cup \{s, x, y, t\}) \neq \emptyset$, and consider the following two cases separately: when $G - X - s$ is balanced and when not.

Case 2.2.1. When $G - X - s$ is balanced.

Since Y is not 3-contractible in G by Claim 22, there exists an arc $e' = sv' \in \delta_G(s)$ with $v' \in Y$ such that $G - e'$ contains a 2-contractible vertex set $X' \subseteq V \setminus \{s, t\}$ with $v' \in X'$ and $N_{G'}(X') = \{x', y'\}$ for some distinct $x', y' \in V \setminus \{s, v'\}$ (recall that, if $G - e'$ contains no 2-contractible vertex set, then we can reduce this case to Case 1 by choosing e' instead of e_0). Choose minimal X'. If $\{x', y'\} \subseteq Y \cup \{x, y, t\}$, then $G[\![X']\!]$ is balanced and hence X' is 3-contractible in G, a contradiction. Besides, if $\{x', y'\} \subseteq X$, then $G - e'$ contains a smaller 2-contractible vertex set $X'' \subsetneq X'$ with $v' \in X''$ and $N_{G'}(X'') = \{x, y\}$.

Thus we have $|\{x', y'\} \cap X| = 1$, and assume $x' \in X$ and $y' \in Y \cup \{x, y, t\}$ (see Fig. 37). Let $Z \subseteq Y$ be the vertex set of the connected component of $G - \{x, y, y'\} - e'$ that contains v'. Then, since Z is not 3-contractible in G and v' is separated from both s and t by $\{x', y'\}$ in $G - e'$, we have $N_{G-e'}(Z) = \{x, y, y'\}$ and $y' \notin \{x, y\}$. If $y' = t$, then this case reduces to Case 2.1 by choosing e' instead of e_0. Otherwise, $\{s, y'\}$ is a 2-cut in G separating v' from t, which contradicts Claim 21.

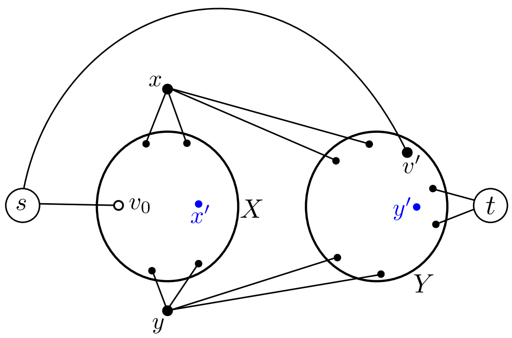

Figure 37: Case 2.2.1.

Case 2.2.2. When $G - X - s$ is not balanced.

Recall that $G[X]$ is connected (discussed just before starting Case 2.1). Suppose that $G[\![X]\!] - x$ and $G[\![X]\!] - y$ are balanced. Then, by Lemma 4, we may assume that $\psi_G(e) = 1_\Gamma$ for every $e \in E(G[\![X]\!])$ by shifting at each $v \in X \cup \{x, y\}$ if necessary. This implies that $G[\![X]\!]$ is also balanced, which contradicts Claim 22.

Thus, at least one of $G[\![X]\!] - x$ and $G[\![X]\!] - y$ is not balanced. By Claim 25 and the symmetry of x and y, $(G[\![X]\!] - x, s, y) \in \mathcal{D}$ and $(G[\![X]\!] - y, s, x) \in \mathcal{D}$. Hence, we may assume that $|l(G[\![X]\!] - y; s, x)| \geq 2$ by Lemma 5. Note that $G - X - s - y$ contains an x–t path (otherwise, $\{s, y\}$ is a 2-cut in G separating x from t, which contradicts Claim 21). This implies $|l(G[\![X]\!] - y; s, x)| = 2$ since $|l(G; s, t)| = 2$.

Let $Z \subseteq Y \cup \{x, y, t\}$ be the set of vertices that are contained in some x–t path in $G - X - s$ (i.e., the vertex set of the 2-connected component of $(G - X - s) + r + rx + rt$ that contains both of x and t, except for r, by Lemma 2). Then, $(G[Z], x, t) \in \mathcal{D}$. If $G[Z]$ is not balanced, then $|l(G[Z]; x, t)| \geq 2$ by Lemma 5, and hence we derive $|l(G; s, t)| \geq 3$ from $|l(G[\![X]\!] - y; s, x)| = 2$ by Lemma 20 (note that there exist $\alpha', \beta' \in l(G[Z]; x, t)$ such that $\alpha' \beta'^{-1} \neq \beta' \alpha'^{-1}$ by Lemma 19). Hence, we assume that $G[Z]$ is balanced, which implies that $Z \neq Y \cup \{x, y, t\}$ (note that $G[Y \cup \{x, y, t\}] = G - X - s$).

Case 2.2.2.1. Suppose that $y \in Z$. Let $W := Y \setminus Z \neq \emptyset$. Since $G[Z] + r + rx + rt$ is a 2-connected component of $G - X - s + r + rx + rt$, we have $|N_{G-s}(W)| \leq 1$ (see Fig. 38). This implies that W is 2-contractible in G, which contradicts Claim 21.

Case 2.2.2.2. Suppose that $Z = Y \cup \{x, t\}$. Note that $G - X - s - x$ contains a y–t path since $\{s, x\}$ is not a 2-cut in G. Hence, G contains no arc between x and y, and there uniquely exists a neighbor $z \in N_{G-X-s}(y)$ with $z \neq x$. Recall that $G - X - s$ is not balanced, which implies that there are parallel arcs between y and z (see Fig. 39). By the definition of Z, $G[Z] - x$ contains a z–t path (possibly of length 0). Hence, by Lemma 20, we may assume that $|l(G[\![X]\!]; s, y)| = 1$.

In this case, $X = \{v_0\}$ and $G'[\![X]\!]$ consists of an arc between v_0 and y and two parallel arcs between v_0 and x, by Claim 25. Suppose that there exists an arc e' from s to $z' \in Z$ in G. If $G[Z]$ contains two disjoint paths between $\{z', t\}$ and $\{x, z\}$ (possibly of length 0, e.g., $z' = z$), then we derive $|l(G; s, t)| \geq 3$ by Lemma 20 (e.g., if $G[Z]$ contains disjoint z'–z path and x–t path, then let P_1, P_2 be two s–y paths obtained from by extending the z'–z path using $e' = sz'$ and the parallel arcs between y and z). Otherwise, by Menger's Theorem, $G[Z]$ contains a 1-cut $w \in Z - t$ separating them, which implies that $\{s, w\}$ is a 2-cut in G, contradicting Claim 21.

Thus we have $s \notin N_G(Z)$. Since $G[Z]$ is balanced and contains no contractible vertex set, $Z = \{x, z, t\}$ (note that $z \notin \{x, t\}$ since $Y \neq \emptyset$). By Lemma 17-(2), there must be single arcs between s and y and between x and t, which leads to Case (B) in Definition 13. Note that the labels of arcs are easily confirmed according to $l(G; s, t) = \{1_\Gamma, \alpha\}$.

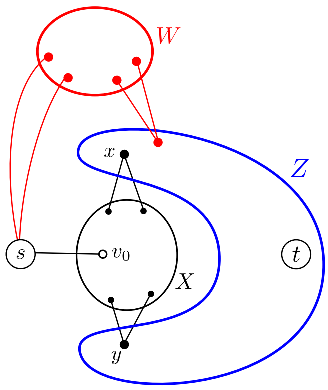

Figure 38: Case 2.2.2.1.

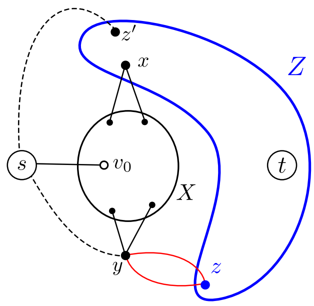

Figure 39: Case 2.2.2.2.

Case 2.2.2.3. Otherwise, $Z \subsetneq Y \cup \{x, t\}$. Let $W := Y \setminus Z \neq \emptyset$. By the definition of Z, we have $N_{G-s}(W) \subseteq \{y, z\}$ for some $z \in Z - x$. Since G contains no 2-contractible vertex set by Claim 21, we have $N_G(W) = \{s, y, z\}$ (see Fig. 40). If $|l(G[\![W]\!] - z; s, y)| \geq 2$, then we derive $|l(G; s, t)| \geq 3$ from $(G'[\![X]\!], x, y) \in \mathcal{D}^1_{\alpha', \beta'}$ by Lemma 20.

Hence, suppose that $|l(G[\![W]\!] - z; s, y)| = 1$. Then, $(G[\![W]\!] - z, s, y) \notin \mathcal{D}$ or $G[\![W]\!] - z$ is balanced by Lemma 5. In the former case, $G[\![W]\!] - z$ contains a 1-cut $w \in W$, which implies that $\{w, z\}$ is a 2-cut in G separating some vertex from both s and t, contradicting Claim 21. In the latter case, there are parallel arcs between z and $w \in W$, since $G[\![W]\!] - s$ is not balanced (recall that $G - X - s$ is not balanced and $G[Z]$ is balanced). If $W \neq \{w\}$, then G contains a contractible vertex set $W' \subsetneq W$ with $N_G(W') \subseteq \{s, w, y\}$ (see Fig. 41), which contradicts Claim 22. Besides, by Claim 25, we have $X = \{v_0\}$ since otherwise we derive $|l(G; s, t)| \geq 3$ from $|l(G[\![X]\!]; s, y)| \geq 2$. Thus, $\{y\}$ is contractible or there are parallel arcs from s to y, which also leads to $|l(G; s, t)| \geq 3$ by Lemma 20.

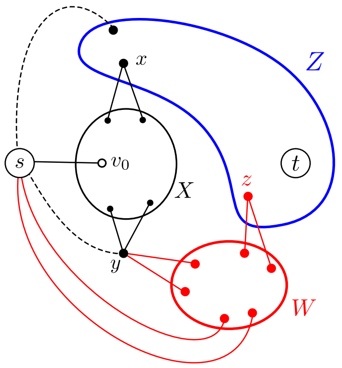

Figure 40: Case 2.2.2.3 (general).

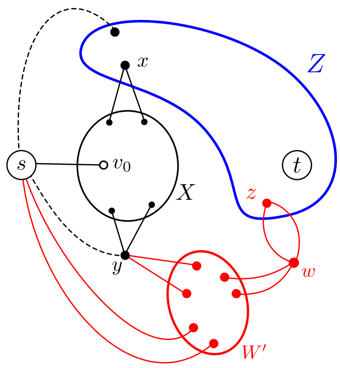

Figure 41: Case 2.2.2.3 (W' is 3-contractible).

Acknowledgments

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