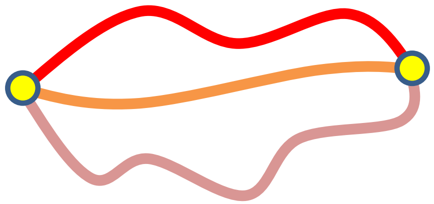
Packing Non-zero A-paths via Linear Matroid Parity

Yutaro Yamaguchi

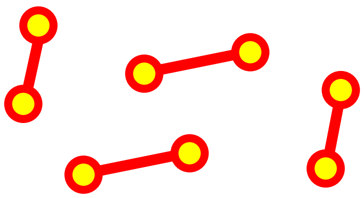
Department of Mathematical Informatics
University of Tokyo

CWCO 2015 @Cargèse September 17, 2015

Overview



Menger's
Disjoint Paths



Non-bipartite
Matching

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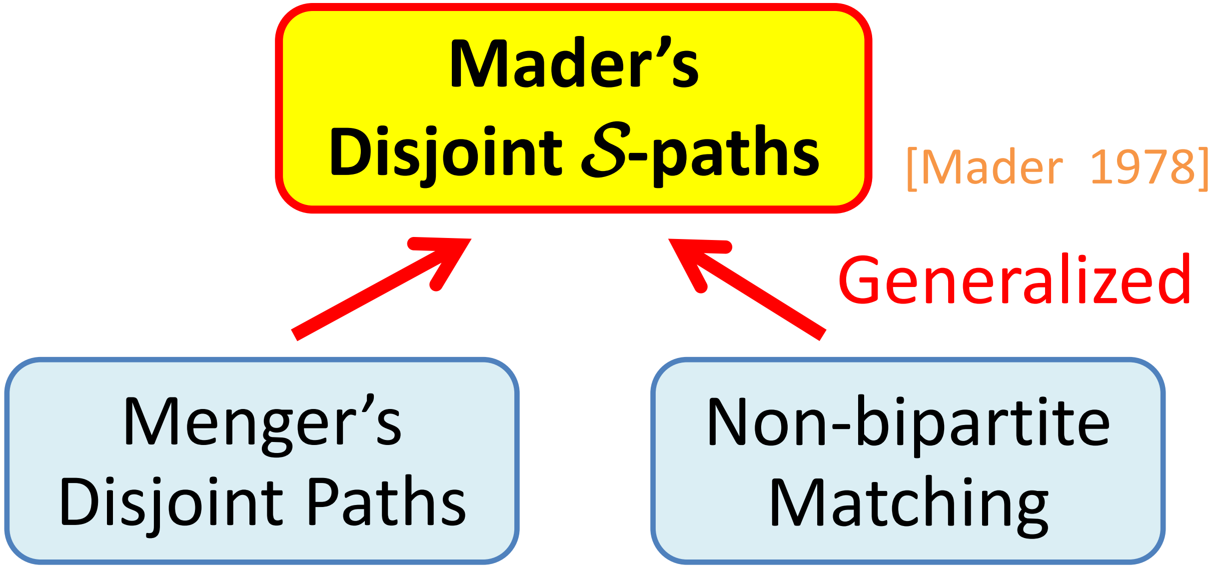
Mader's
Disjoint $\mathcal{S}$-paths
[Mader 1978]
Generalized
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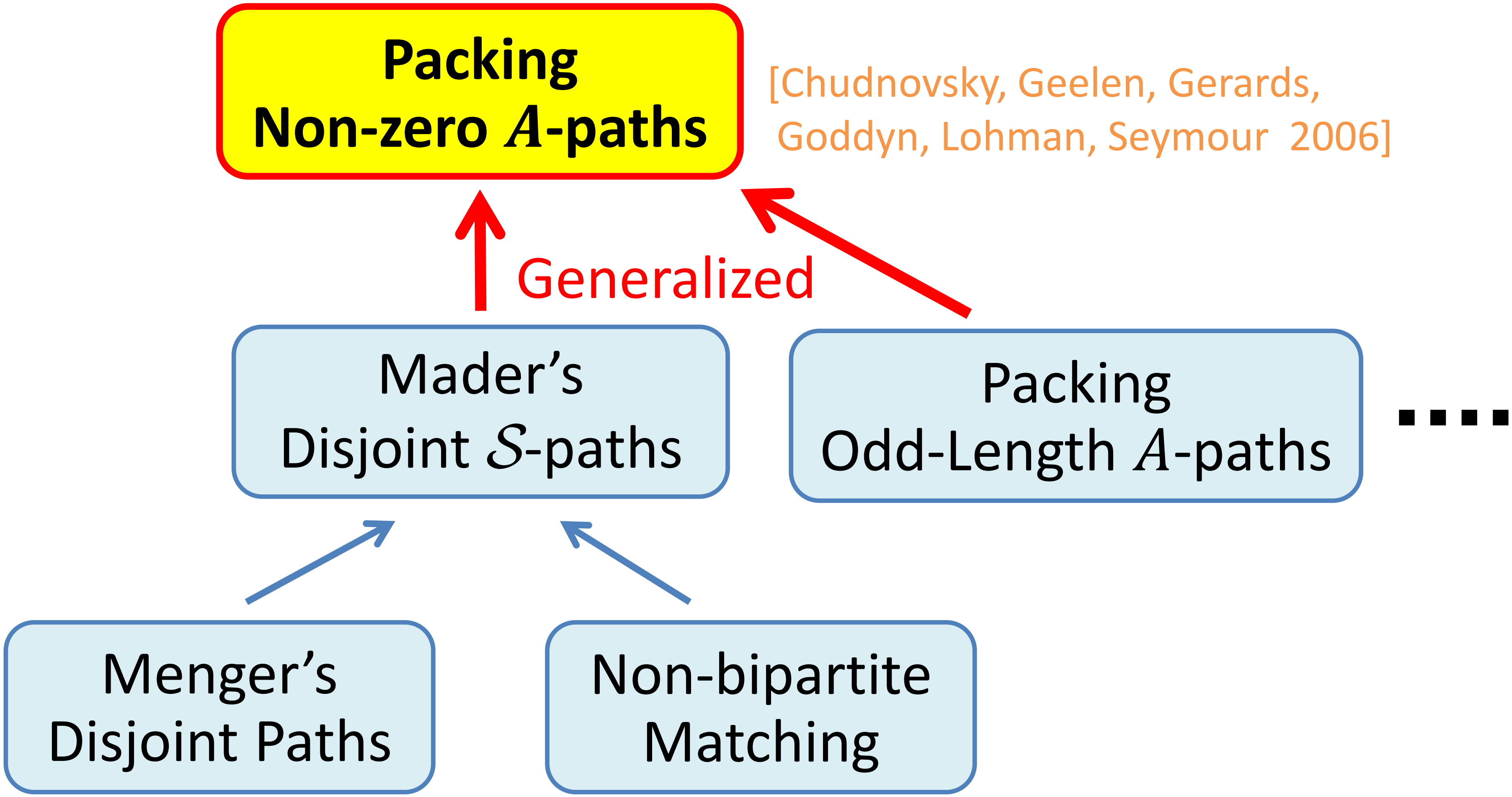
Packing
Non-zero A-paths
[Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]
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Odd-Length A-paths
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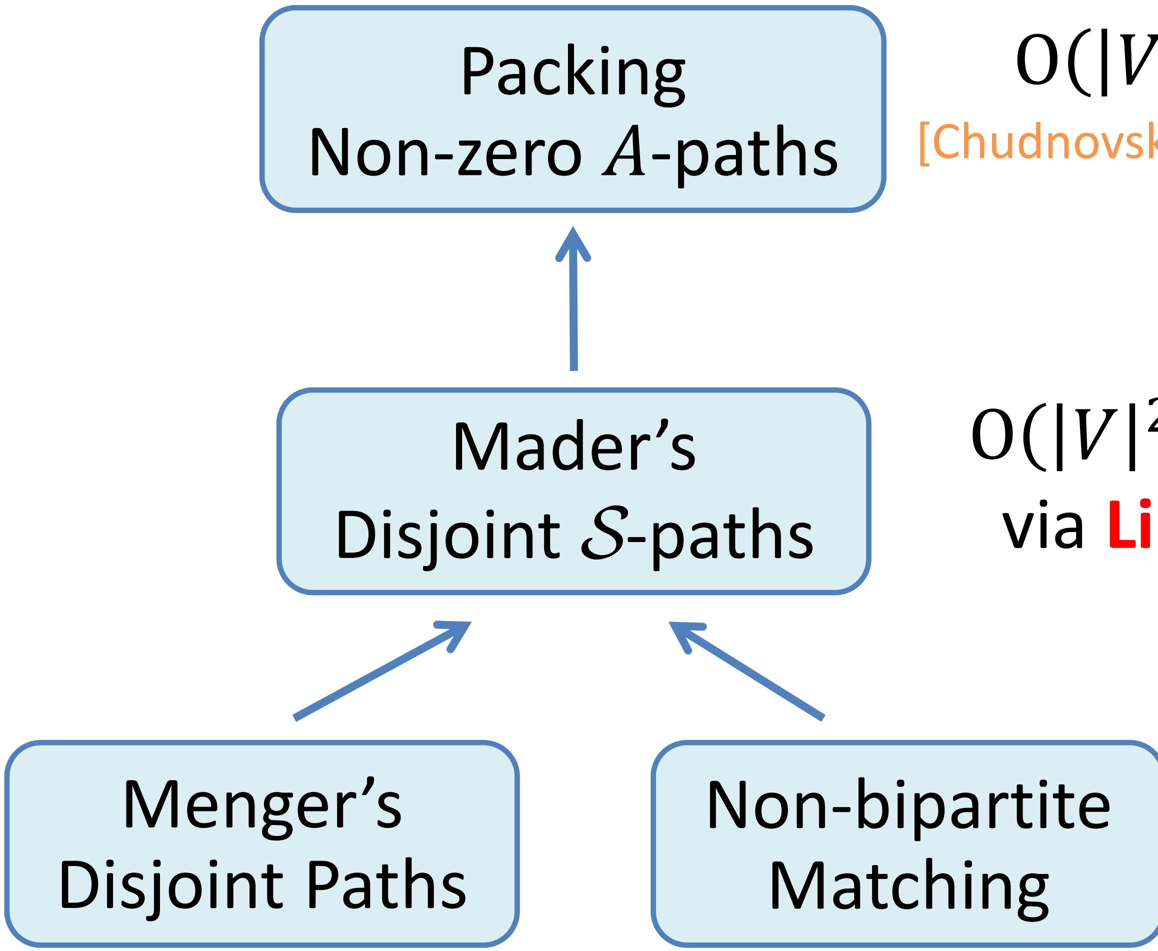

$O(|V|^5)$-time Algorithm

[Chudnovsky, Cunningham, Geelen 2008]

$O(|V|^{2.373})$-time Algorithm

via **Linear Matroid Parity**

[Schrijver 2003]

[Cheung, Law, Leung 2014]

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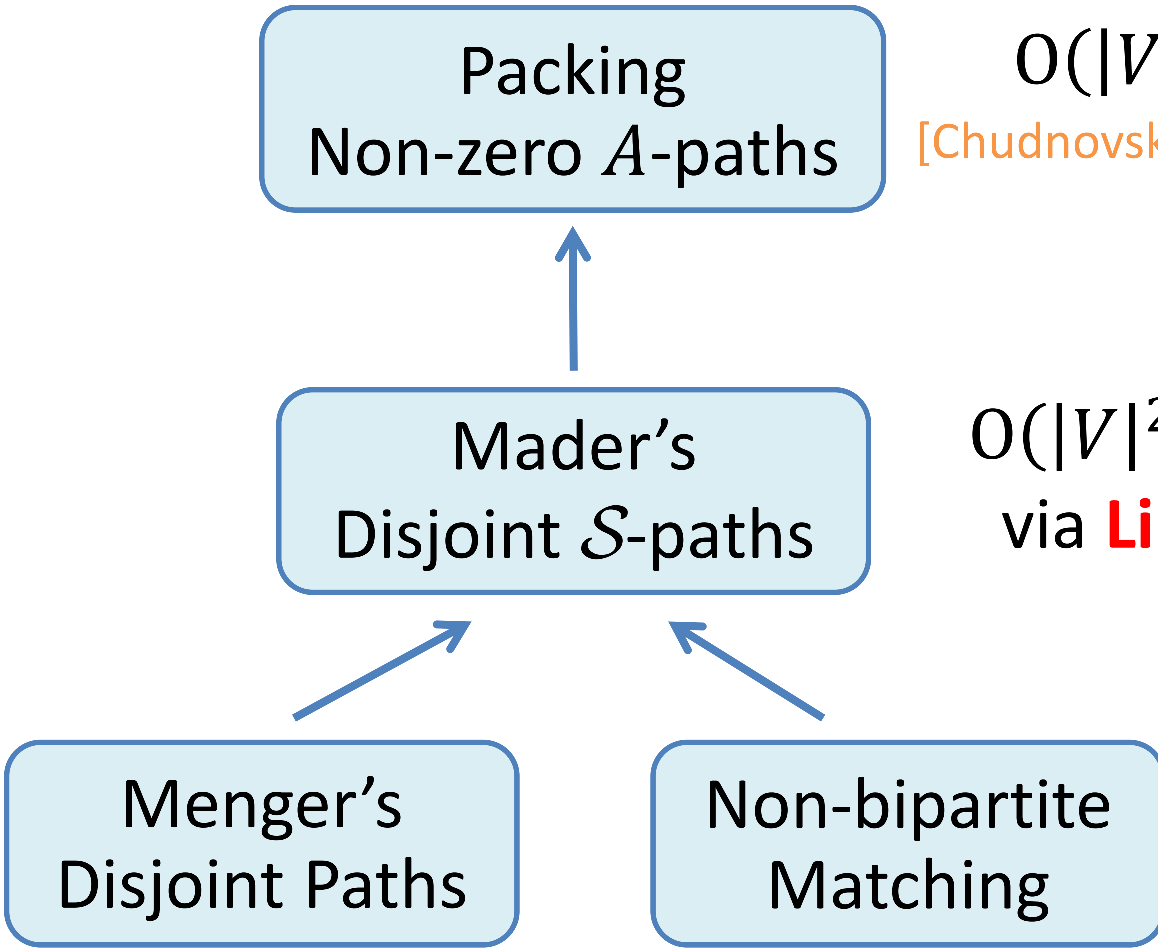

$O(|V|^5)$-time Algorithm
[Chudnovsky, Cunningham, Geelen 2008]

Extendable?

$O(|V|^{2.373})$-time Algorithm
via **Linear Matroid Parity**
[Schrijver 2003]
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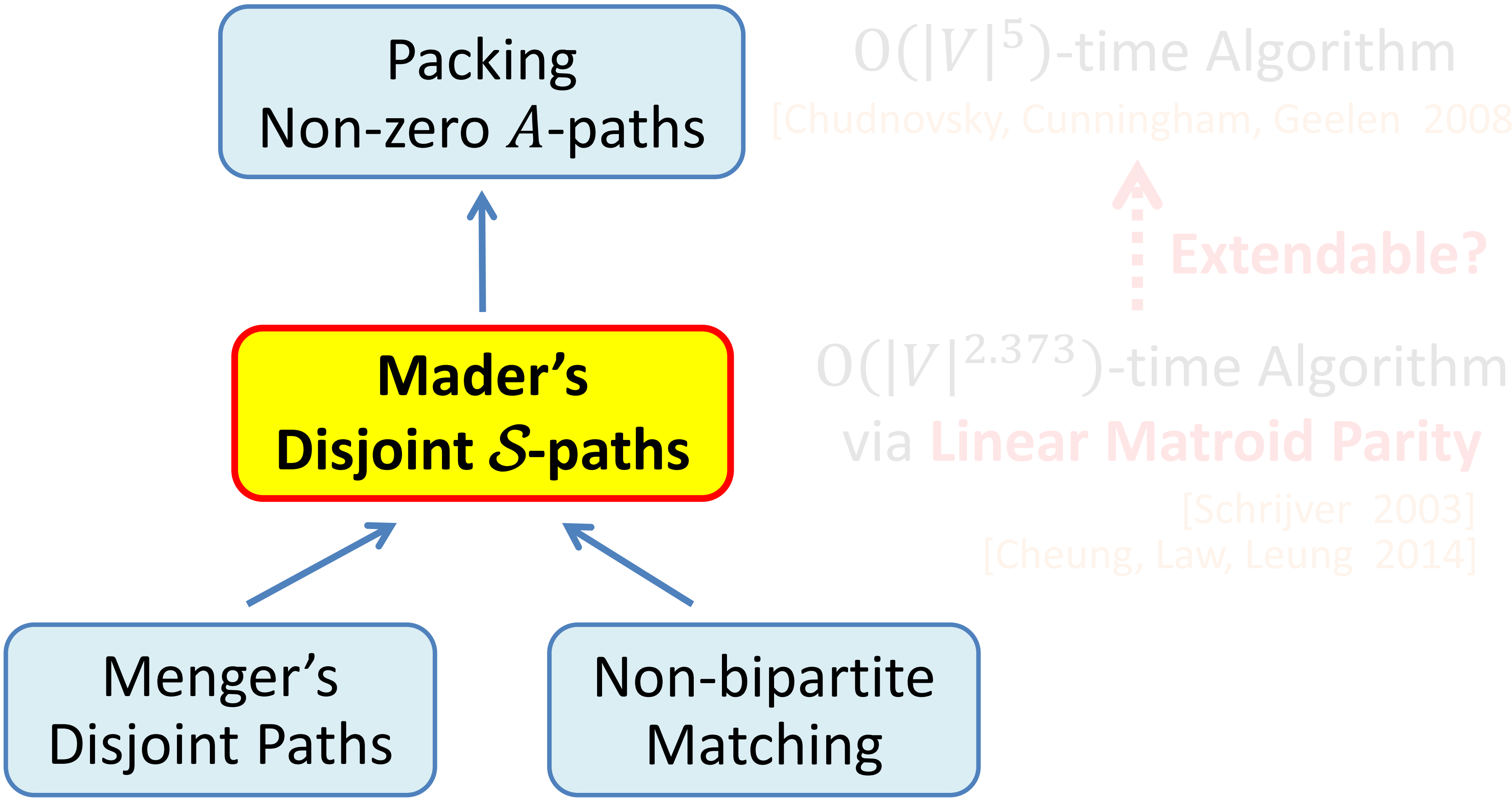
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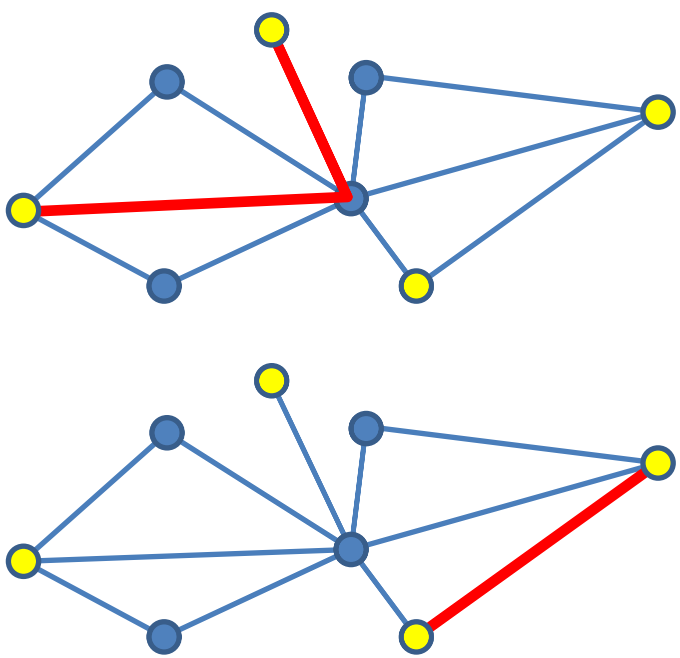
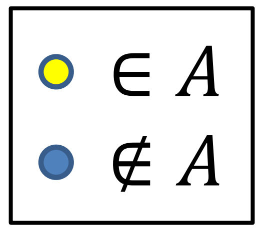
A-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

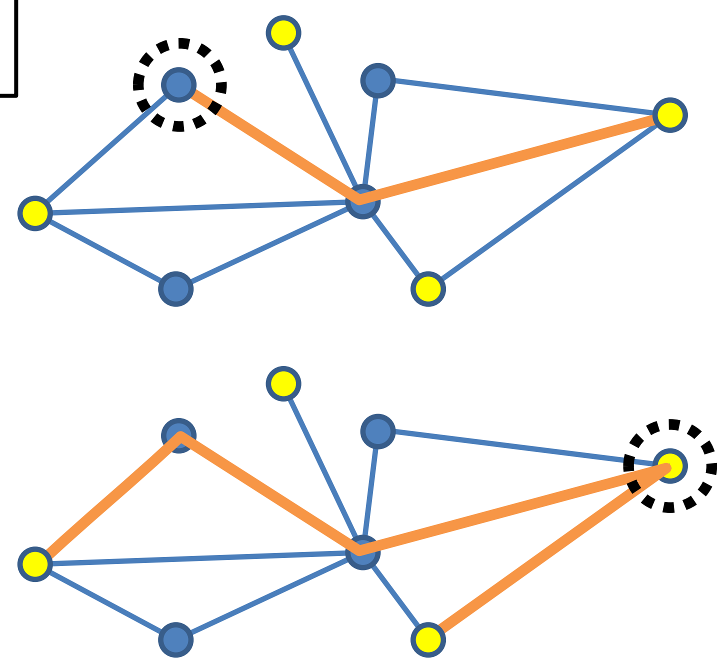
$A \subseteq V$: Terminal Set

A-paths





NOT A-paths

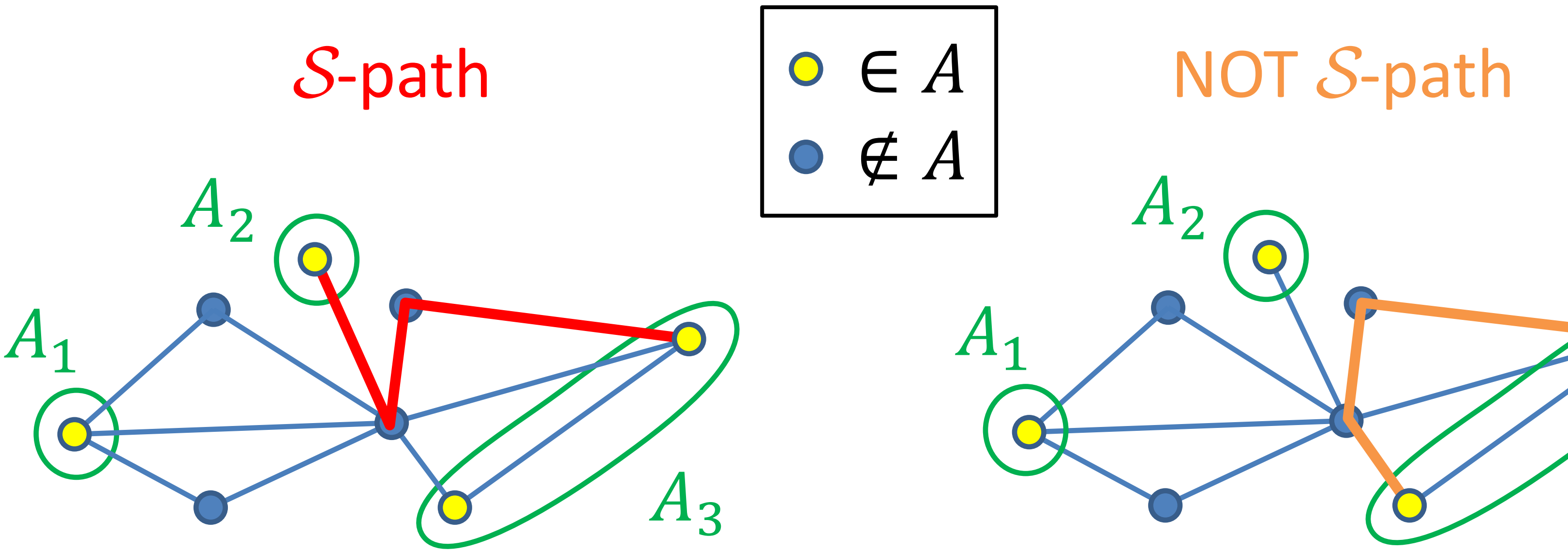


A-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set

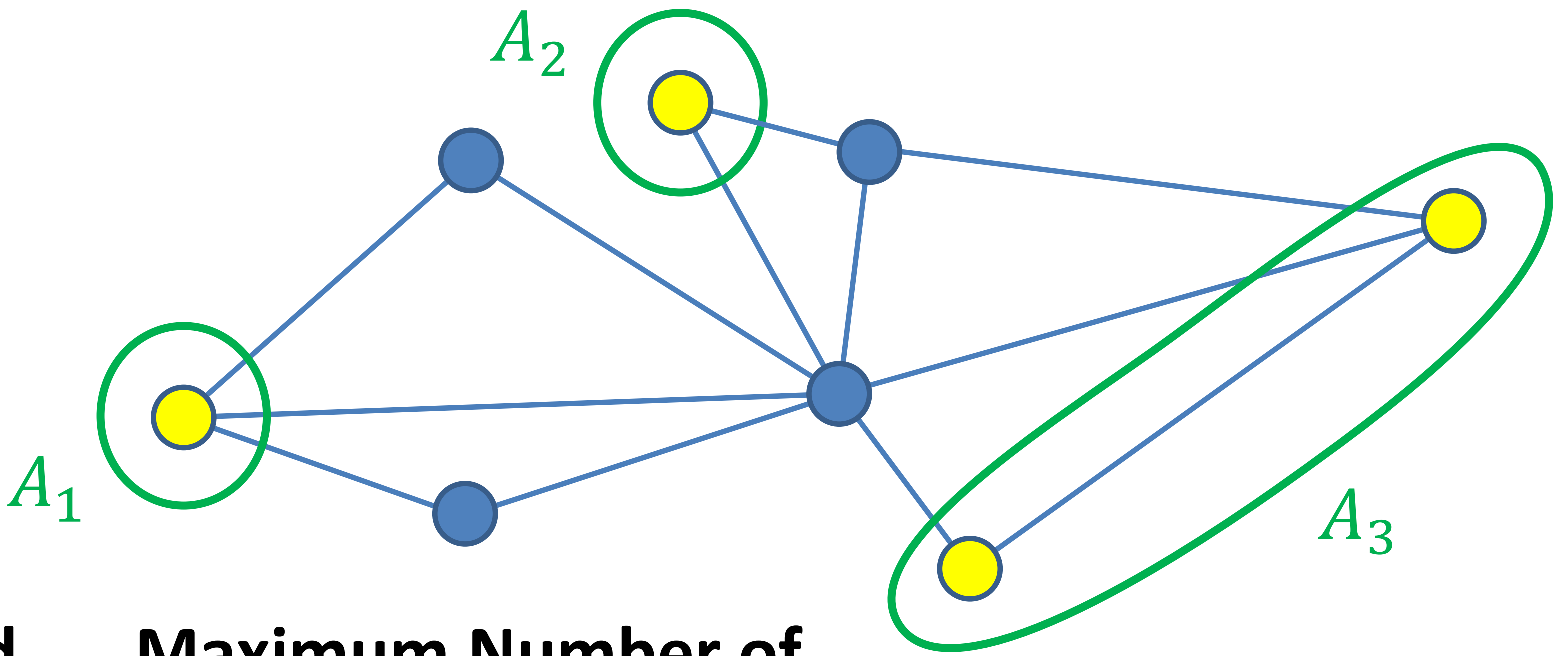
$\mathcal{S} = \{A_1, A_2, \ldots, A_k\}$: **Partition** of A



Mader's Disjoint $\mathcal{S}$-paths Problem

Given $G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of A



Find **Maximum Number of**

Fully Vertex-Disjoint $\mathcal{S}$-paths

Mader's Disjoint $\mathcal{S}$-paths Problem

Given $G = (V, E)$: Undirected Graph

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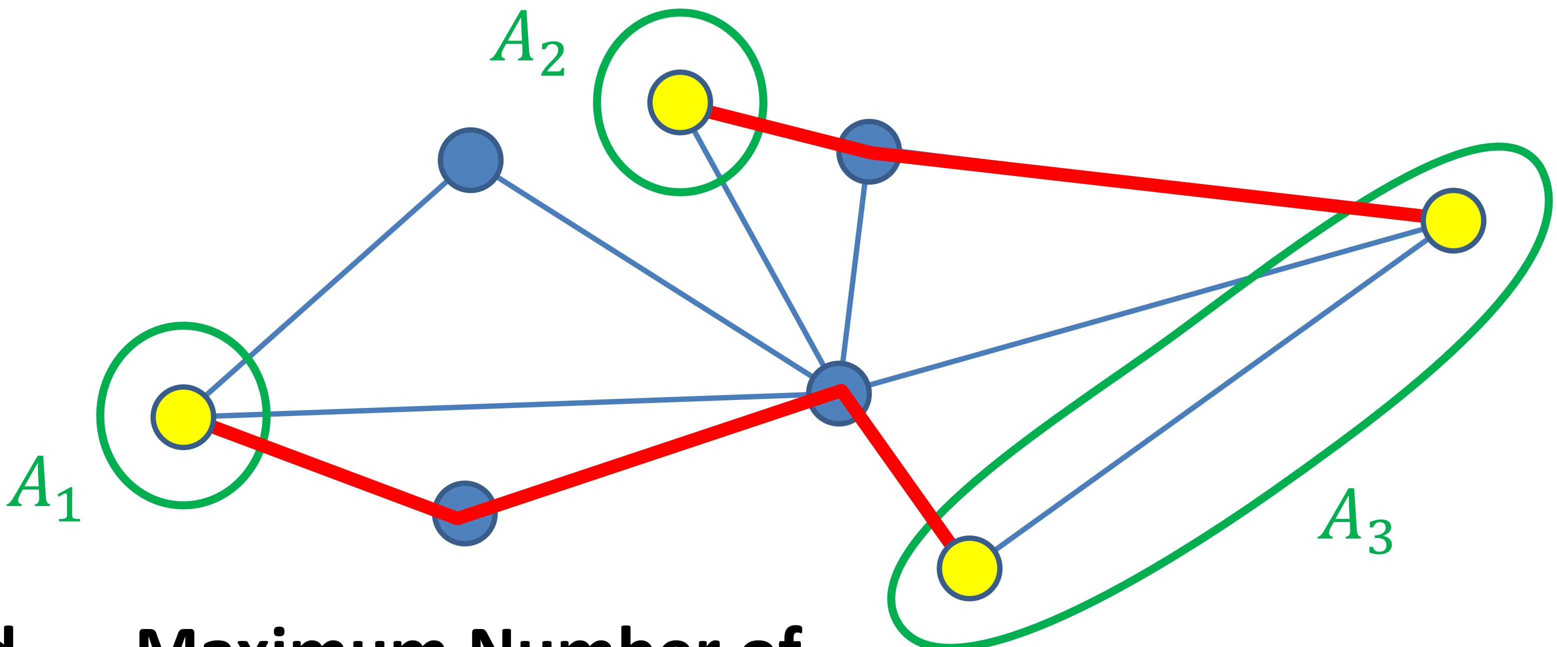

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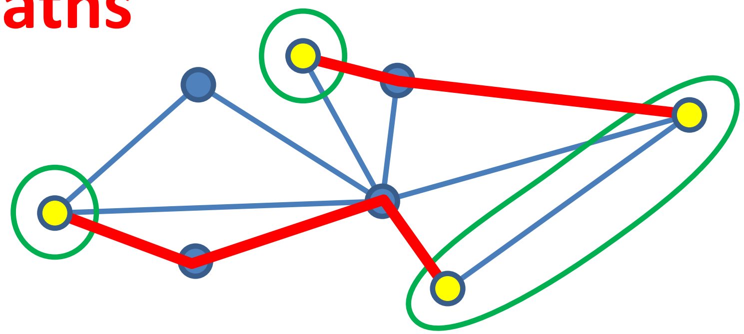

- Min-Max Duality [Mader 1978]
- Polytime via **Matroid Matching** [Lovász 1980, 1981]

Mader's Disjoint $\mathcal{S}$-paths Problem

Find **Maximum Number of**
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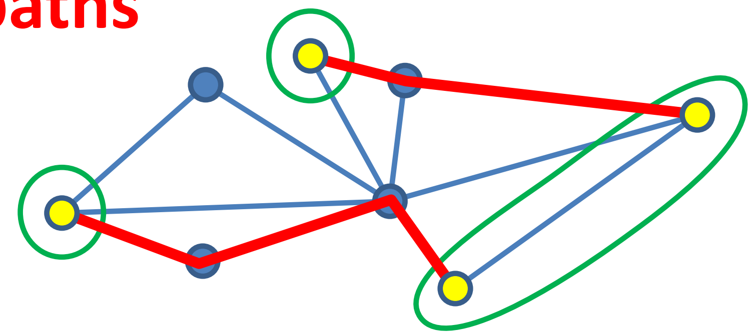
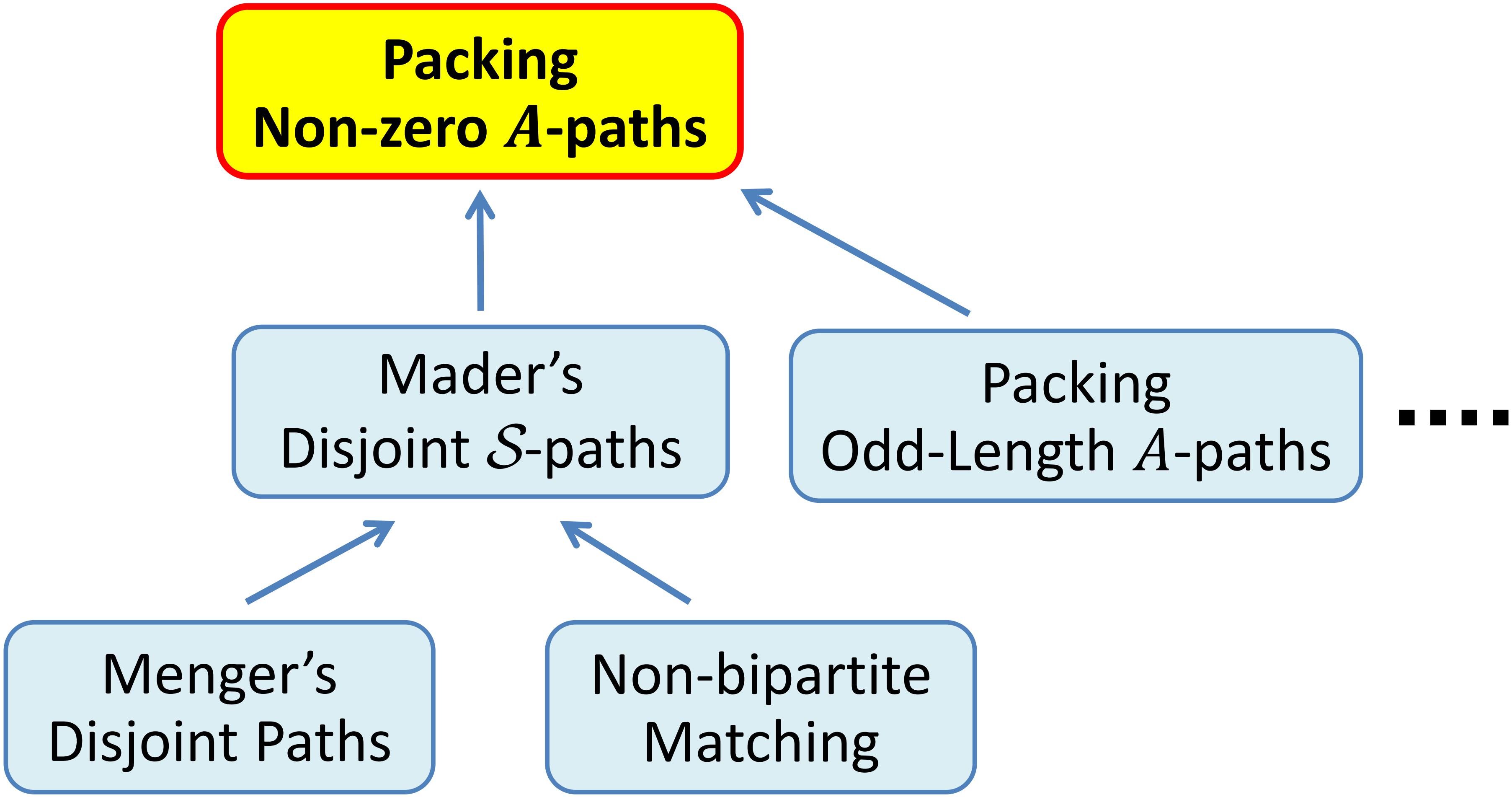


- Min-Max Duality [Mader 1978]
- Polytime via **Matroid Matching** [Lovász 1980, 1981]
  → **Linear Representation** [Schrijver 2003]
  → $O(|V|^{\omega})$-time Algorithm via **Linear Matroid Parity**
  $(\omega < 2.373)$ [Cheung, Law, Leung 2014]

Overview



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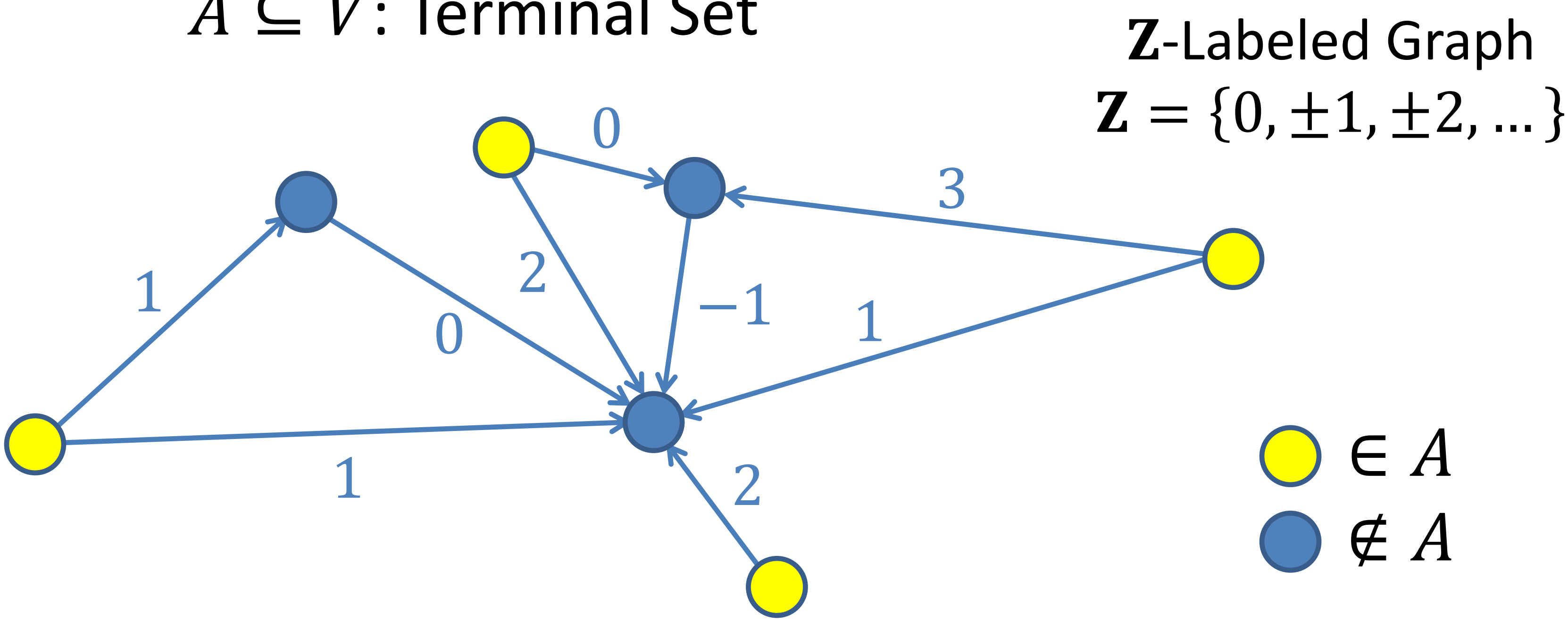
Menger's Disjoint Paths
Non-bipartite Matching

Packing Non-zero A-paths

Given $G = (V, E)$: Group-Labeled Graph

$A \subseteq V$: Terminal Set



Find **Maximum Number of**
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Packing Non-zero A-paths

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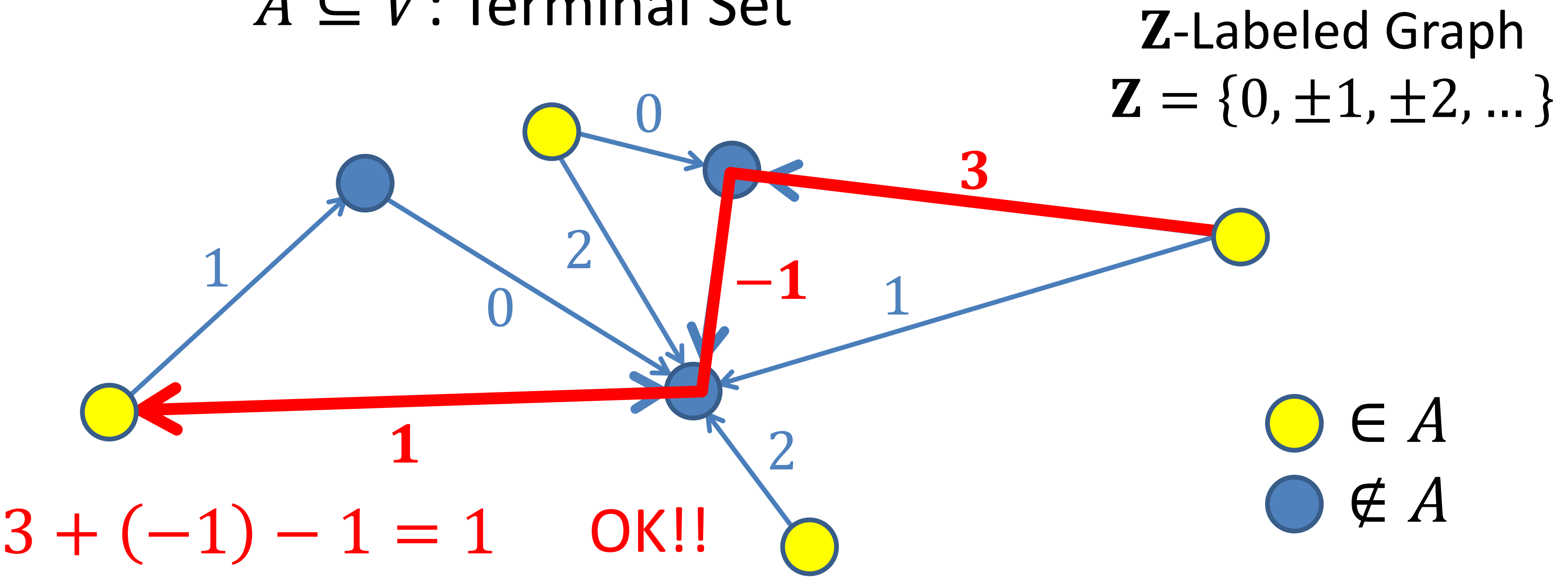

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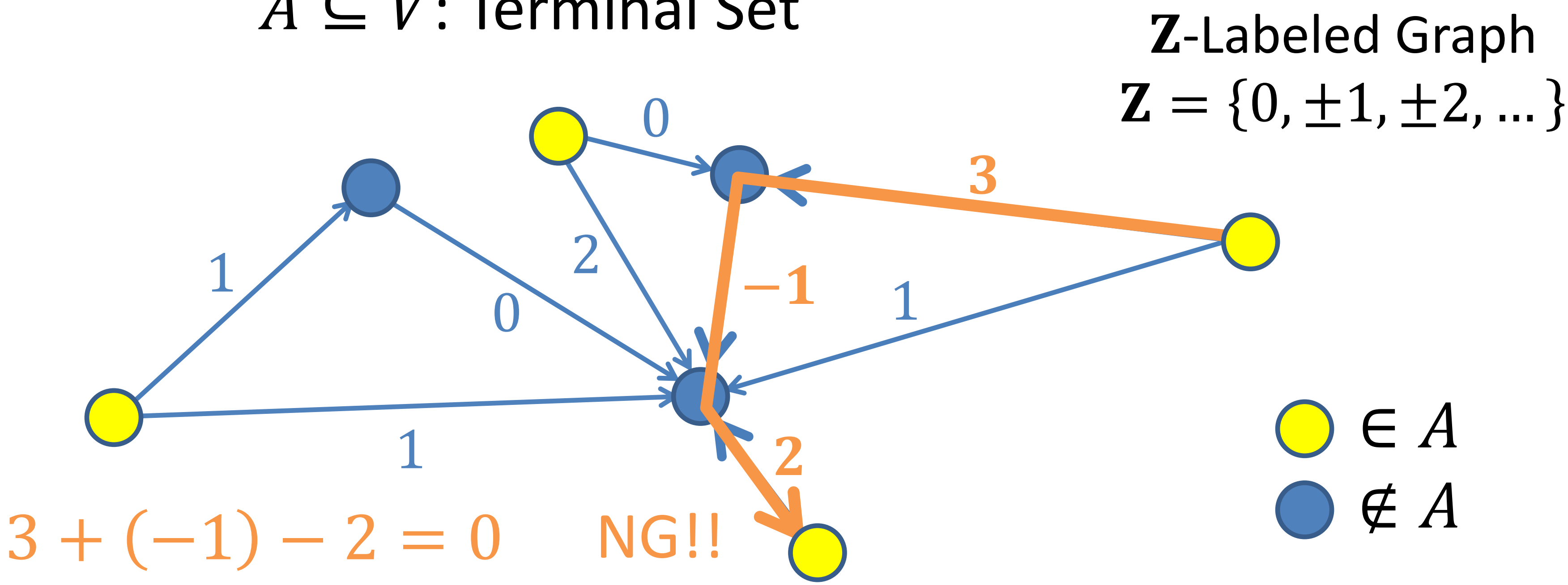

Find **Maximum Number of**

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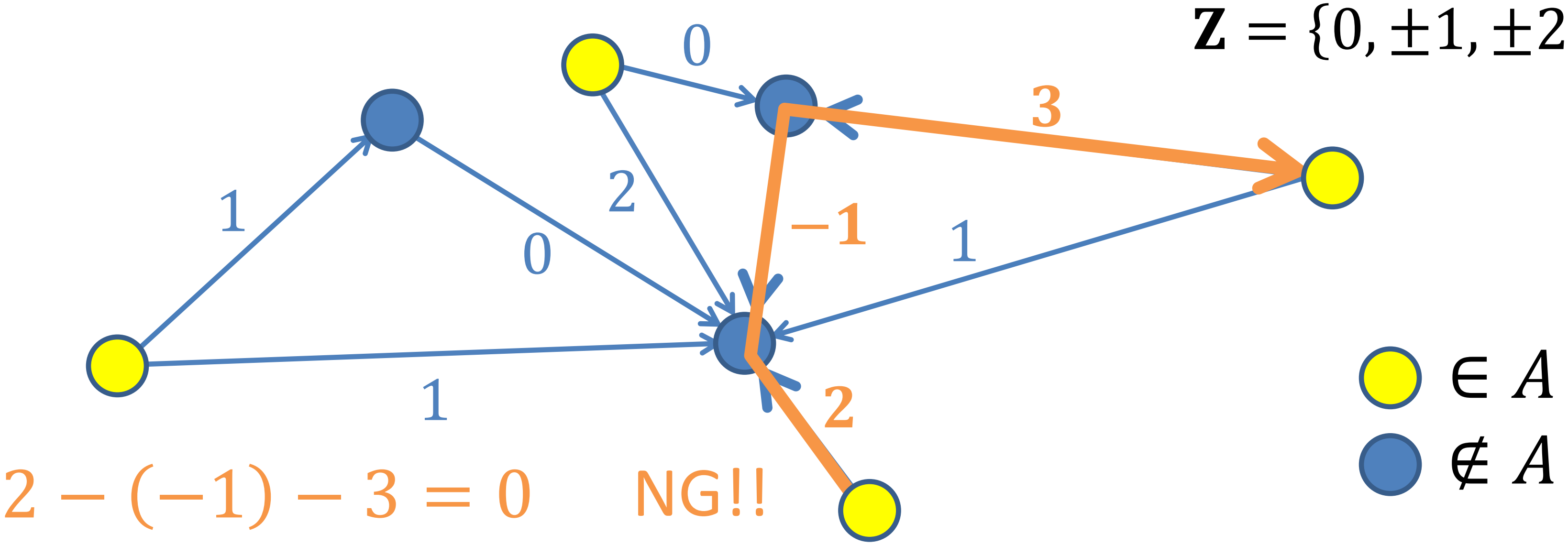
Packing Non-zero A-paths

Given $G = (V, E)$: Group-Labeled Graph

$A \subseteq V$: Terminal Set

$\mathbf{Z}$-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$



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Ex. 1 Mader's $\mathcal{S}$-paths

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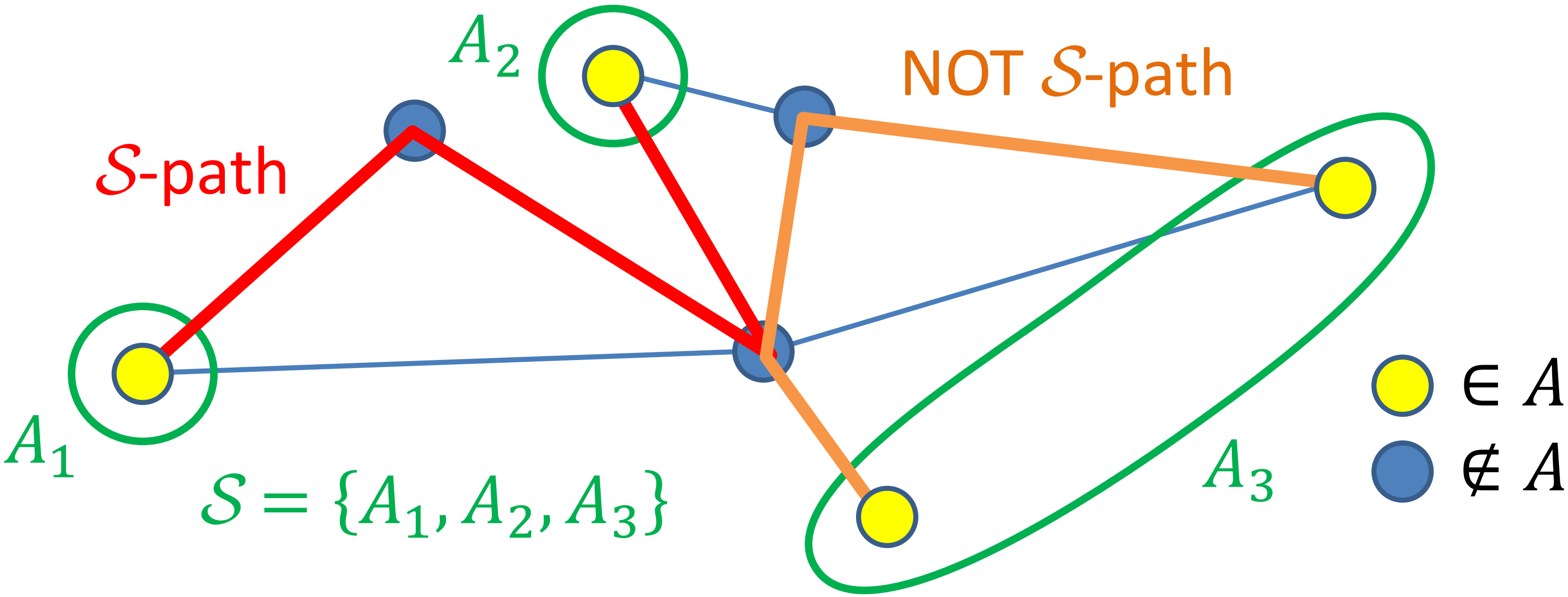

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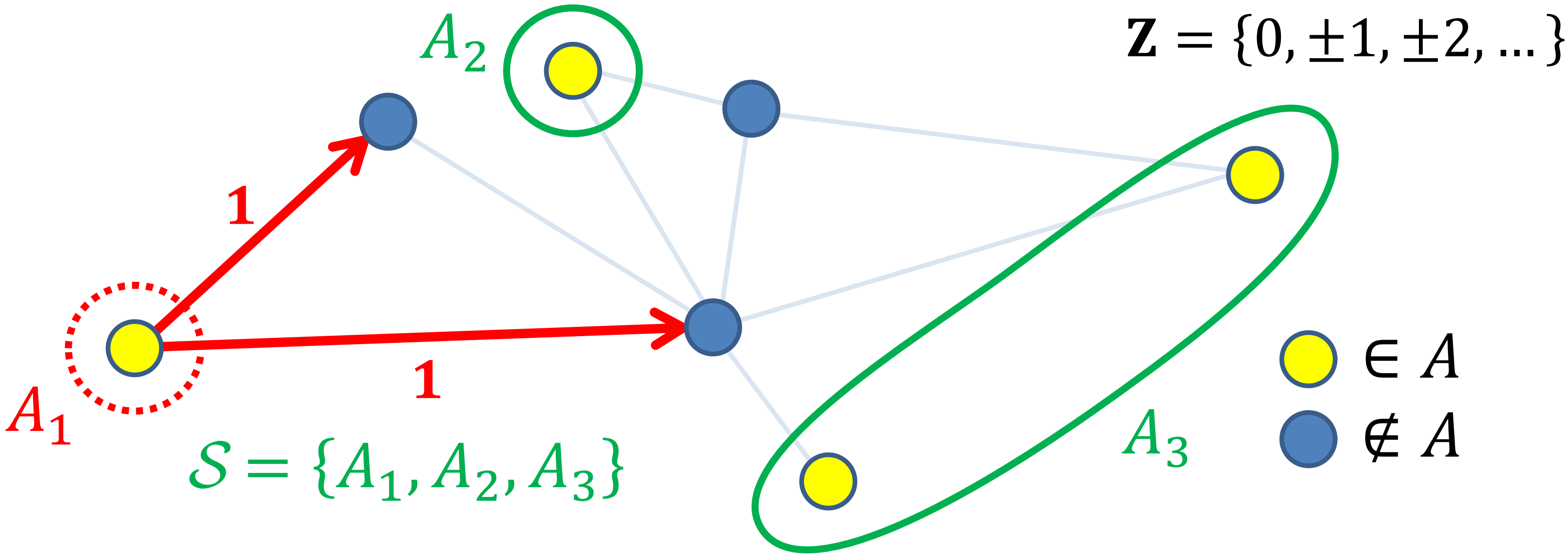

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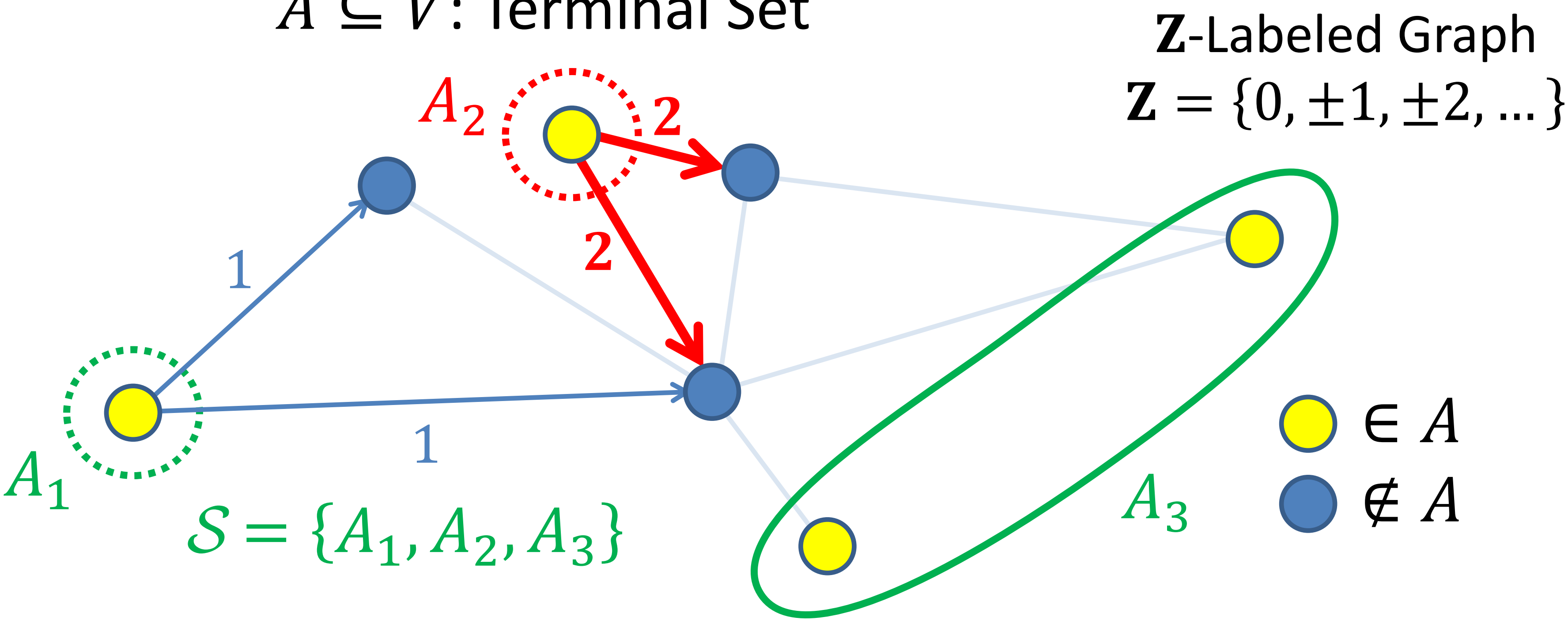

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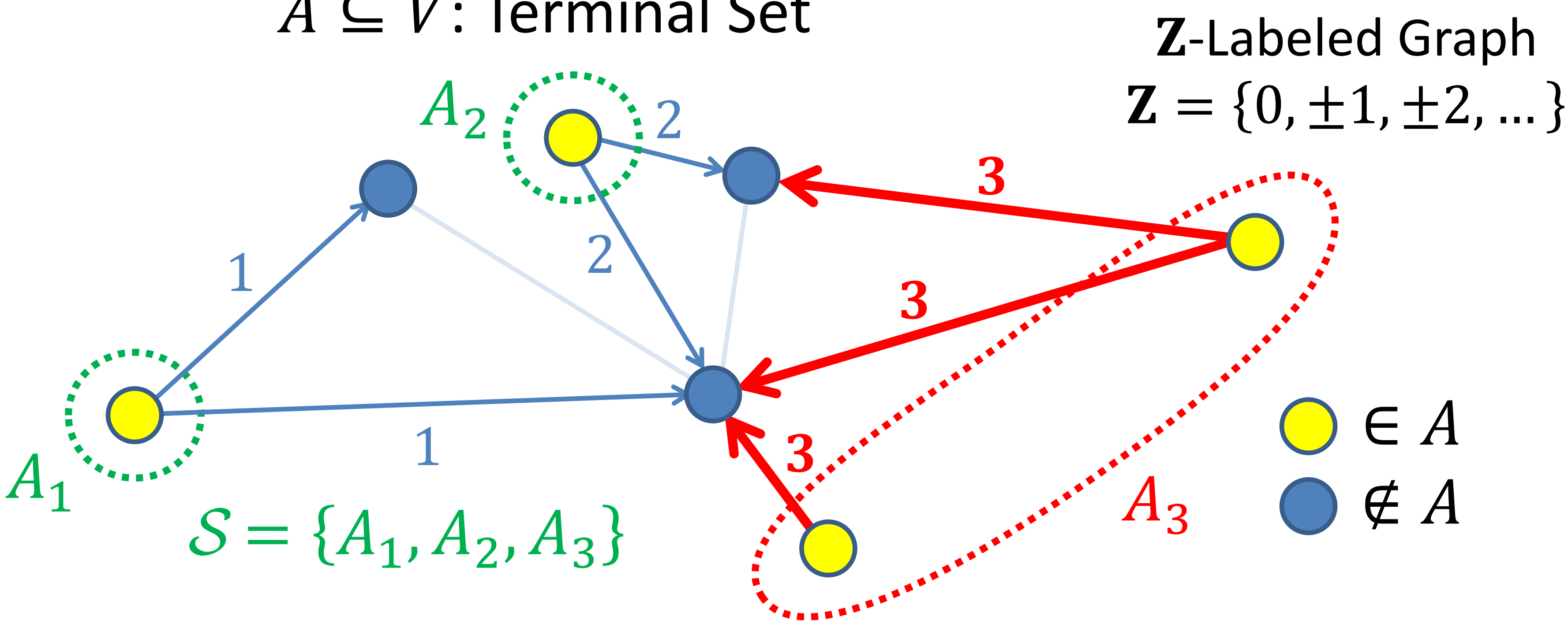

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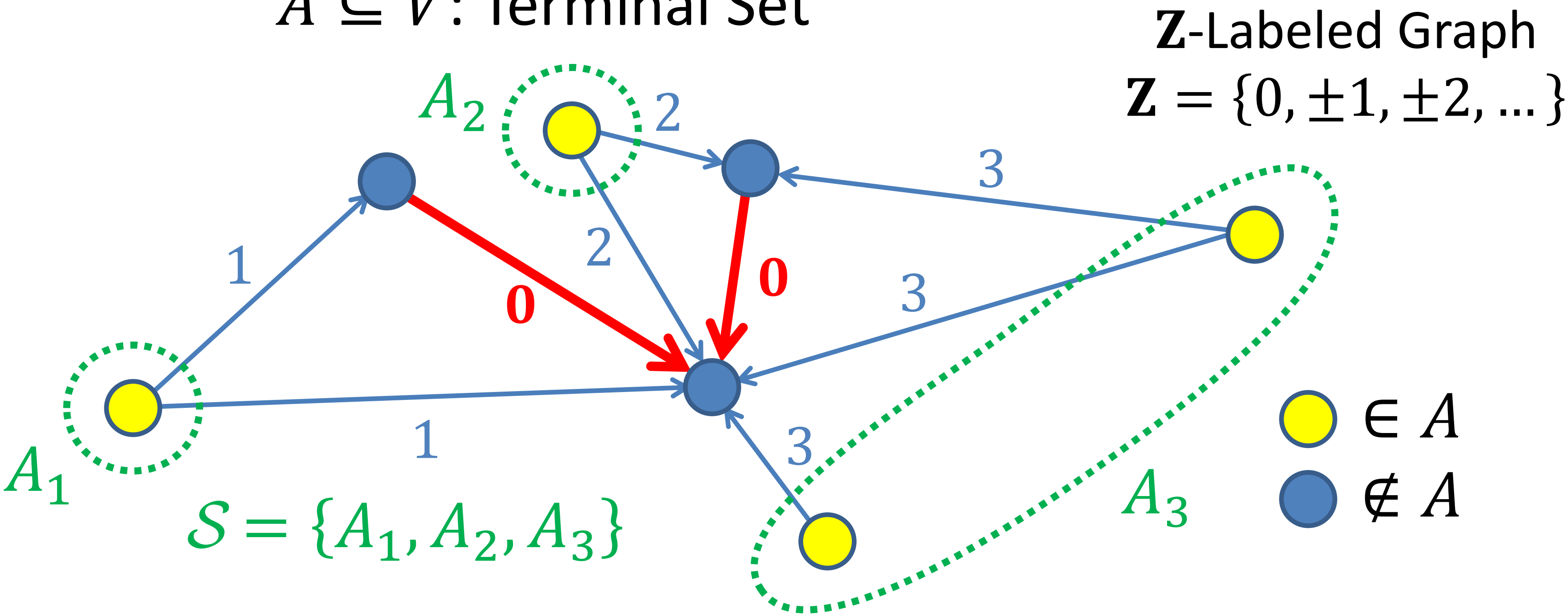

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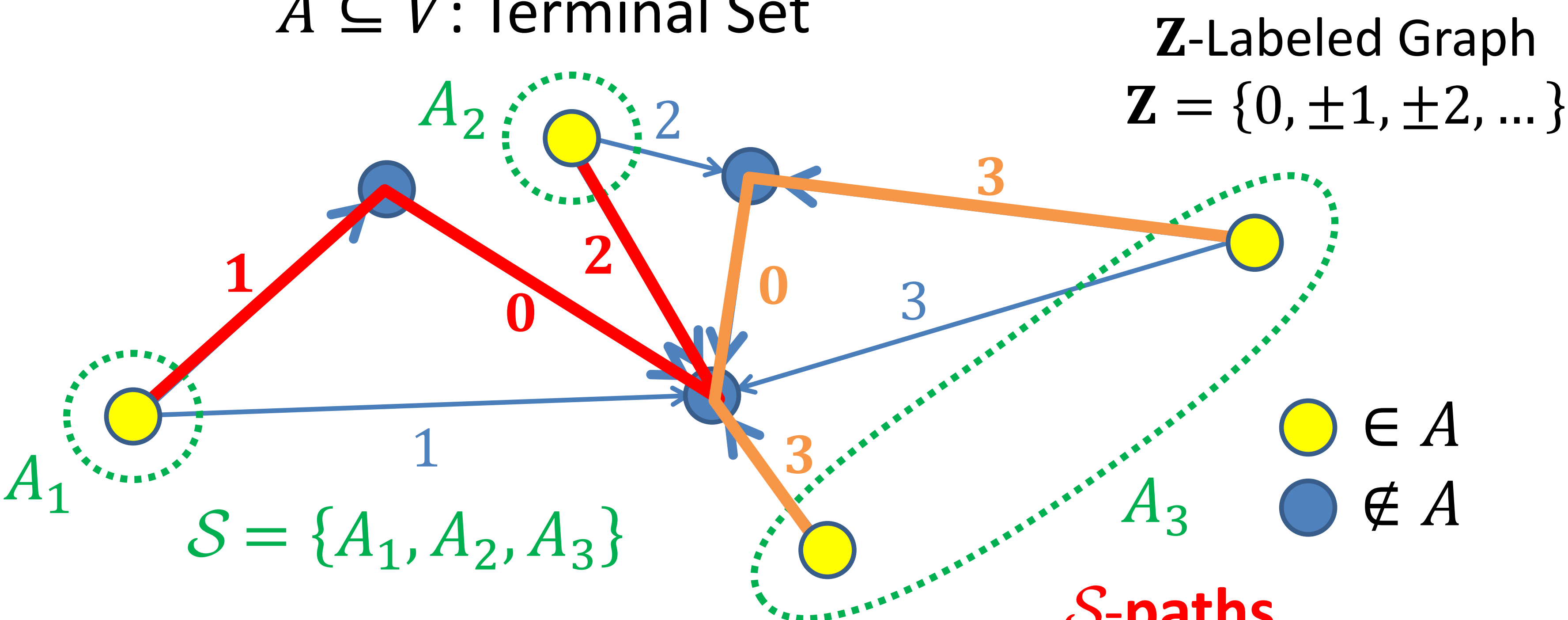


Find **Maximum Number of Fully Vertex-Disjoint Non-zero A-paths**

$\mathcal{S}$-paths ⇕ Non-zero A-paths

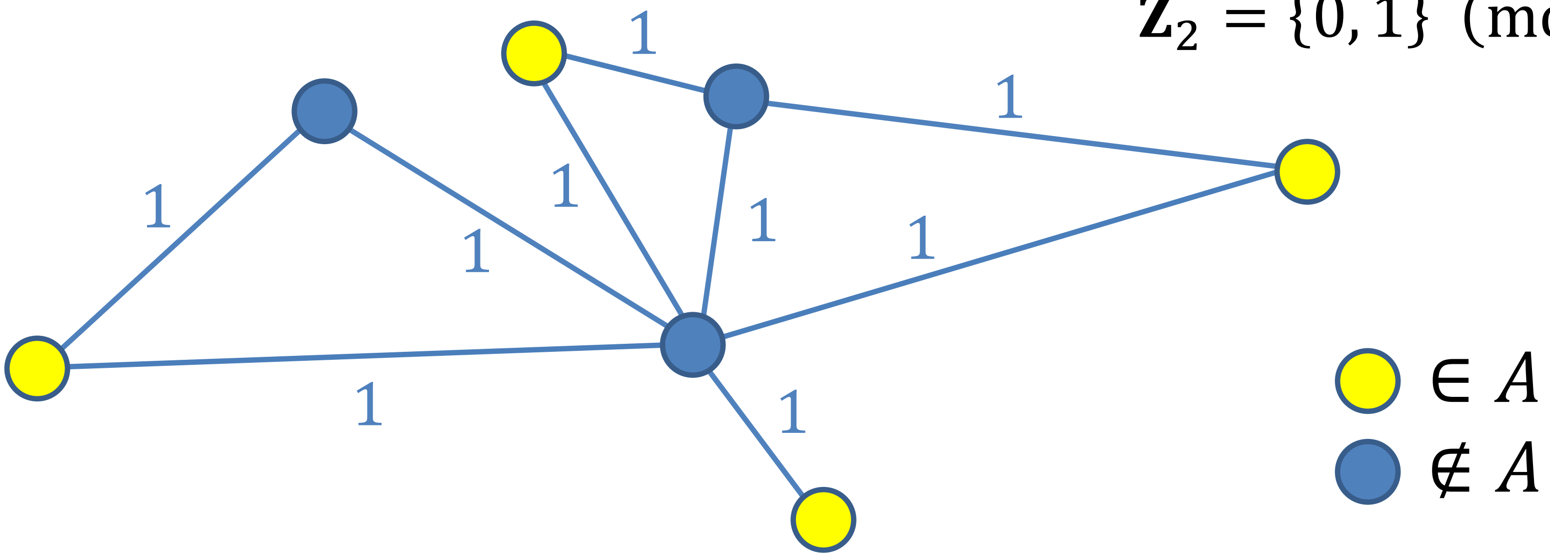
Ex. 2 Odd-Length A-paths

Given $G = (V, E)$: Group-Labeled Graph

$A \subseteq V$: Terminal Set

$\mathbf{Z}_2$-Labeled Graph

$\mathbf{Z}_2 = \{0, 1\} \pmod 2$



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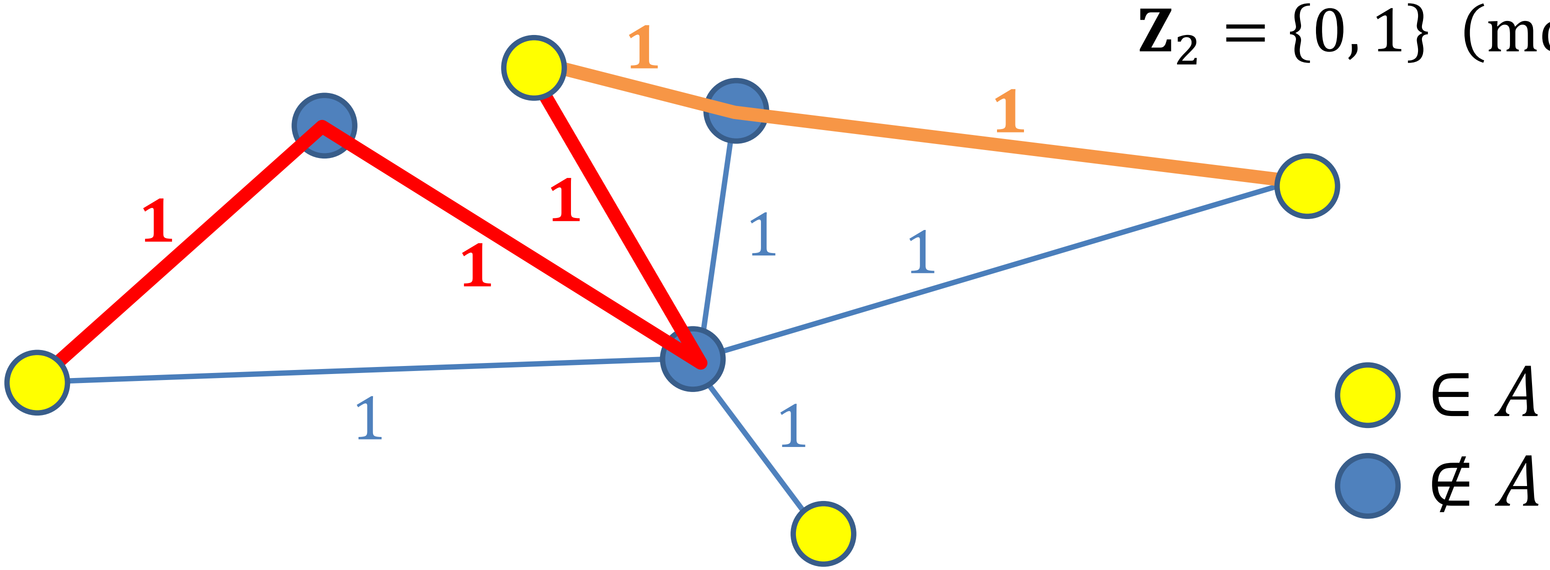
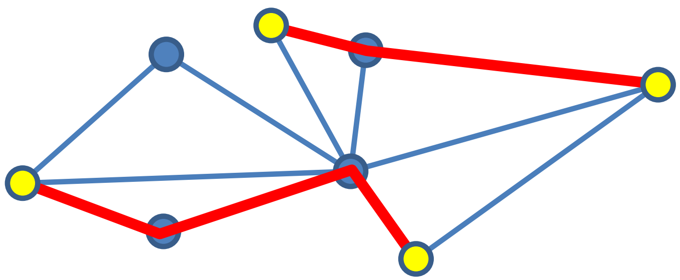


Find **Maximum Number of** **Odd-Length** ⇗⇙ **Fully Vertex-Disjoint Non-zero A-paths**

Packing Non-zero A-paths

<u>Find</u> **Maximum Number of <u>Fully Vertex-Disjoint</u>**
Non-zero A-paths



- Min-Max Duality [Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]
- $O(|V|^5)$-time Algorithm [Chudnovsky, Cunningham, Geelen 2008]

Packing Non-zero A-paths

<u>Find</u> **Maximum Number of <u>Fully Vertex-Disjoint</u>**
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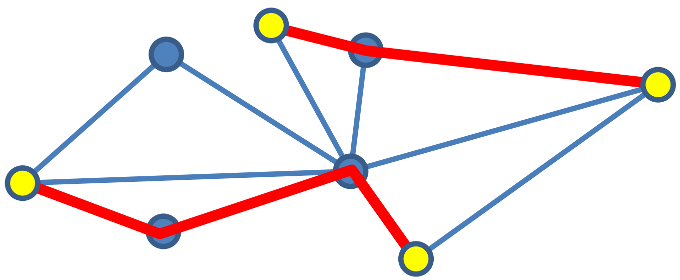

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$\rightarrow$ **Improvable??**

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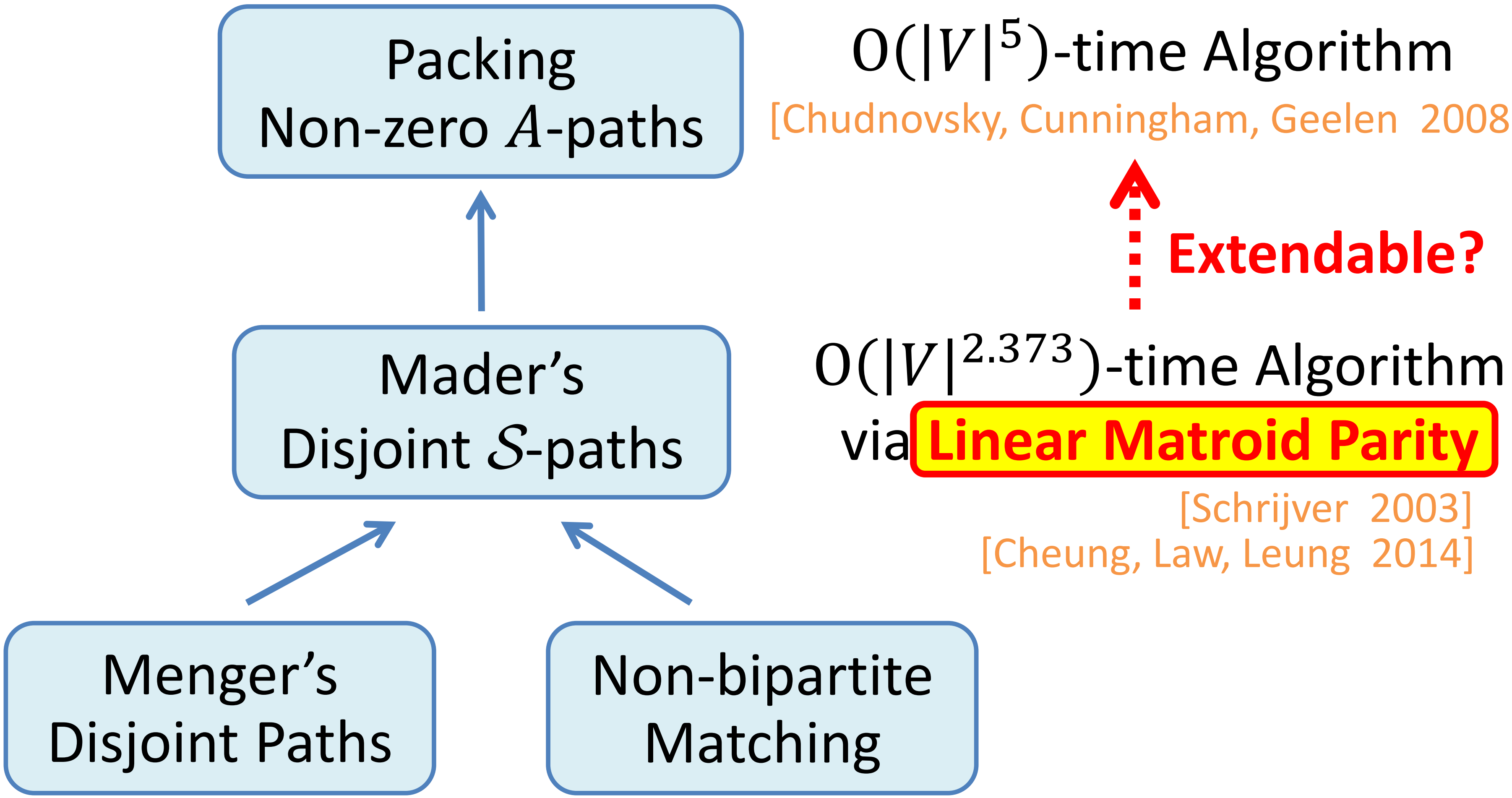

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Linear Matroid Parity Problem

Given $Z \in \mathbf{F}^{n \times 2m}$: Matrix with **Pairing** of Columns

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find **Maximum Number of Linearly Independent Column-Pairs**

Linear Matroid Parity Problem

Given $Z \in \mathbf{F}^{n \times 2m}$: Matrix with **Pairing** of Columns

Full Rank
$(\text{rank} = 6)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Find **Maximum Number of**
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NOT Full Rank $(\mathrm{rank} = 3)$

Find **Maximum Number of Linearly Independent Column-Pairs**

Linear Matroid Parity Problem

Find **Maximum Number of**
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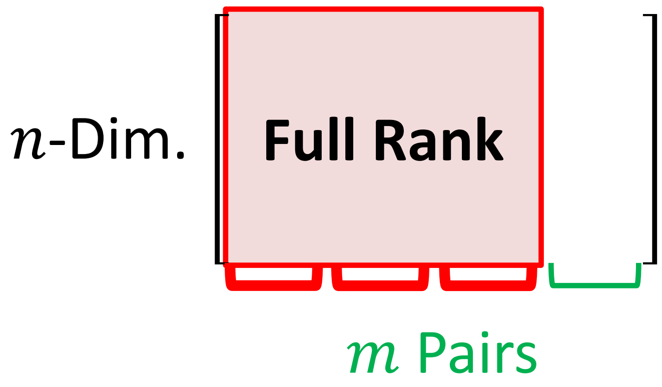


- Min-Max Duality [Lovász 1980]
- **Polytime** Solvable

$O(m^{17})$? (First Polytime) [Lovász 1981]

$O(mn^{\omega})$ (Deterministic) [Gabow, Stallmann 1986]

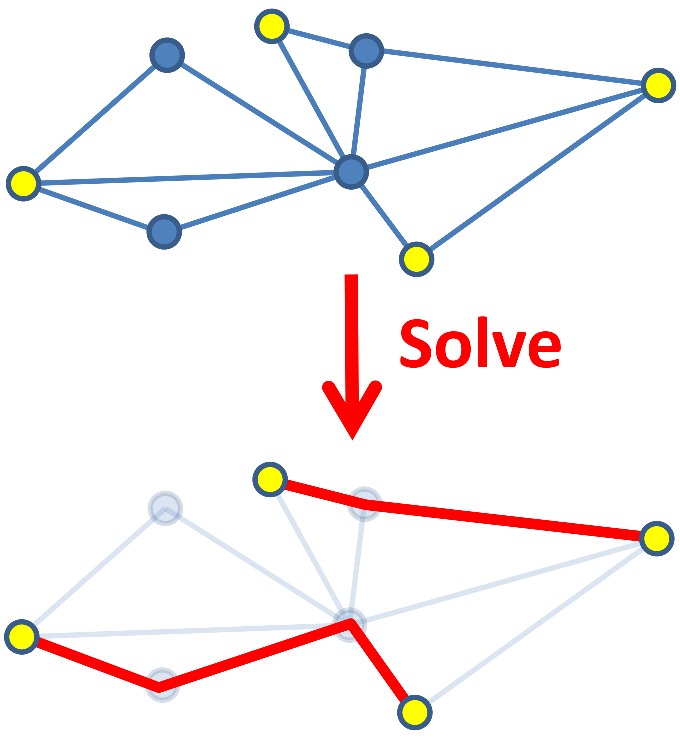
$O(mn^{\omega-1})$ (Randomized) [Cheung, Law, Leung 2014]

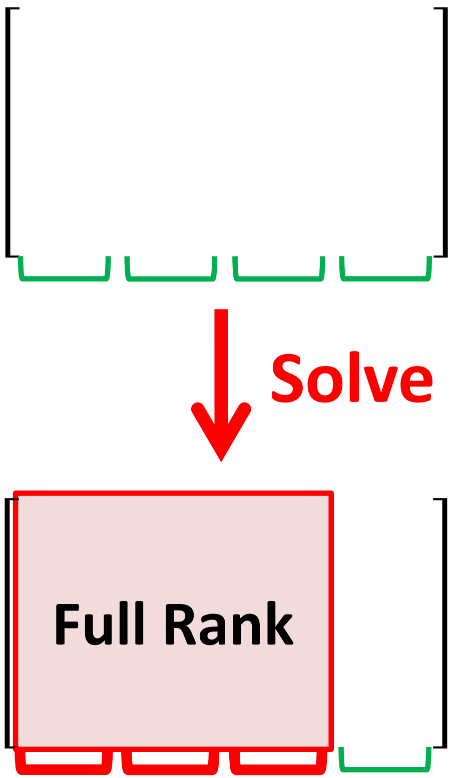
($\omega < 2.373$: Matrix Multiplication Exponent)

Reduction Flow

Packing
Non-zero A-paths

Linear Matroid Parity





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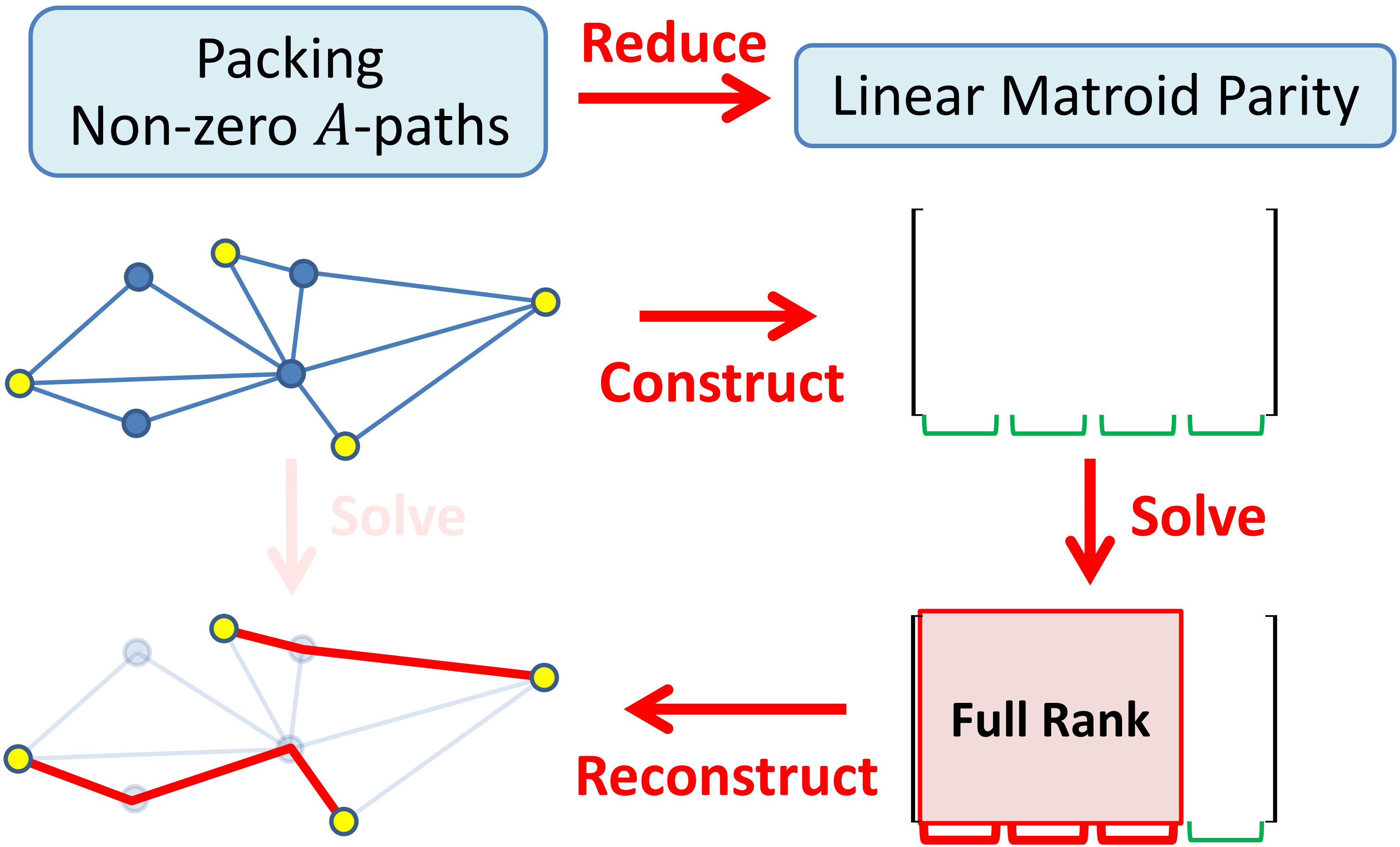

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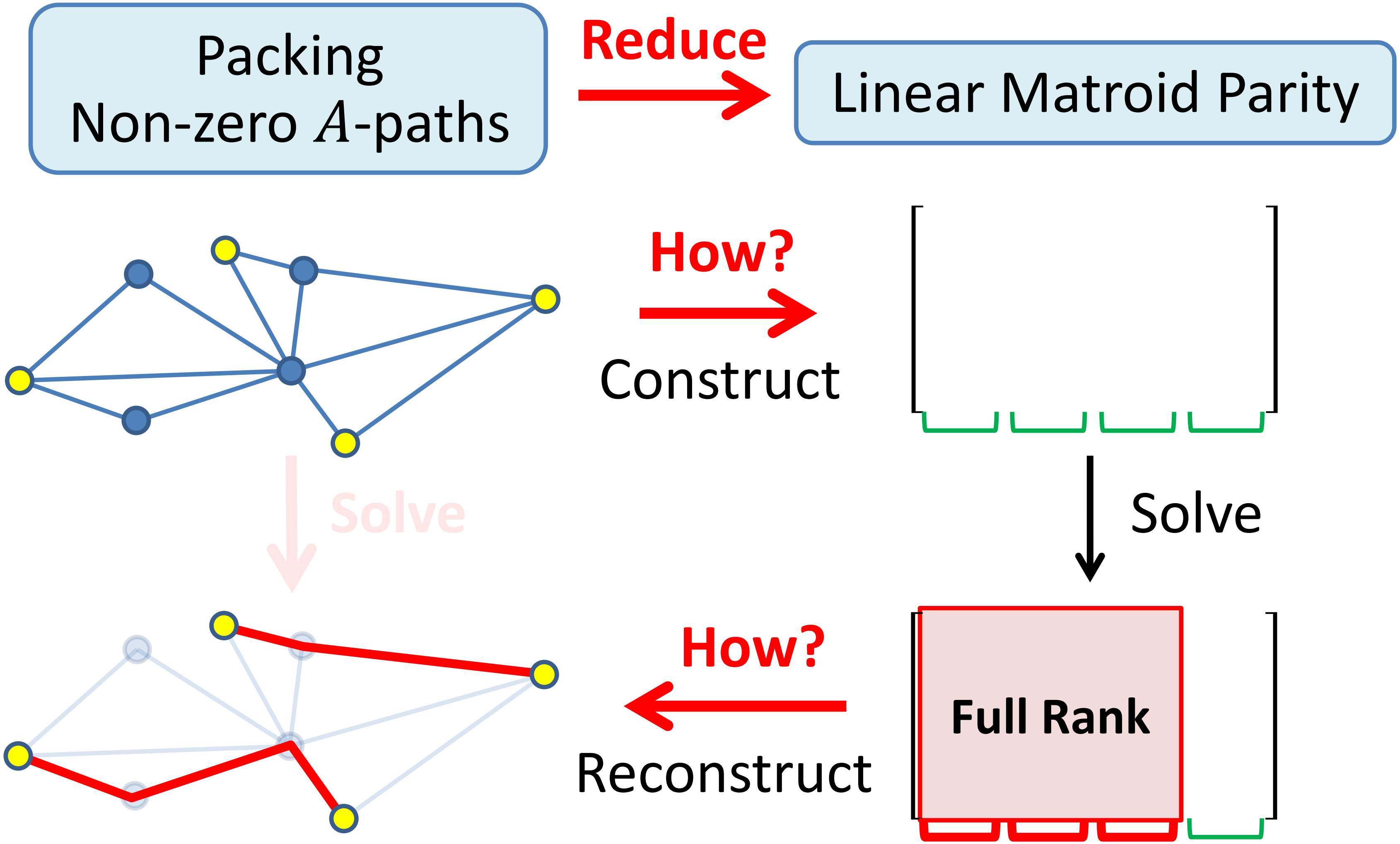
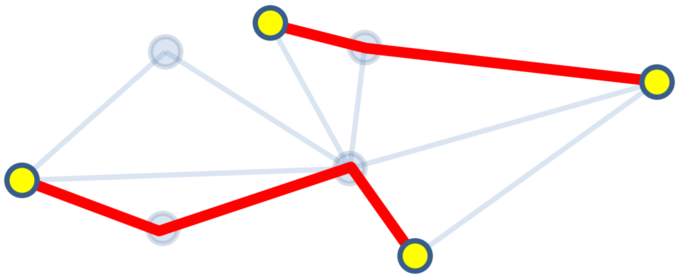
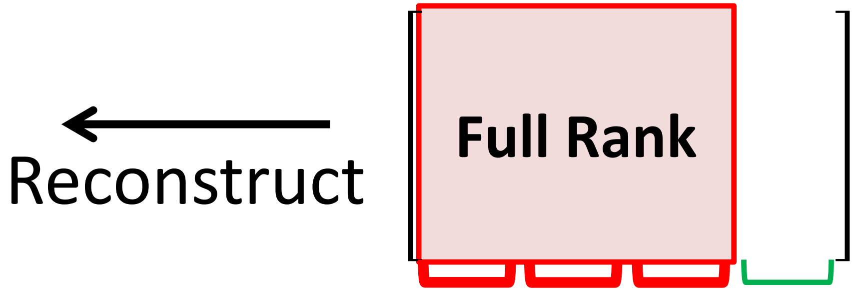


Associated Matrix

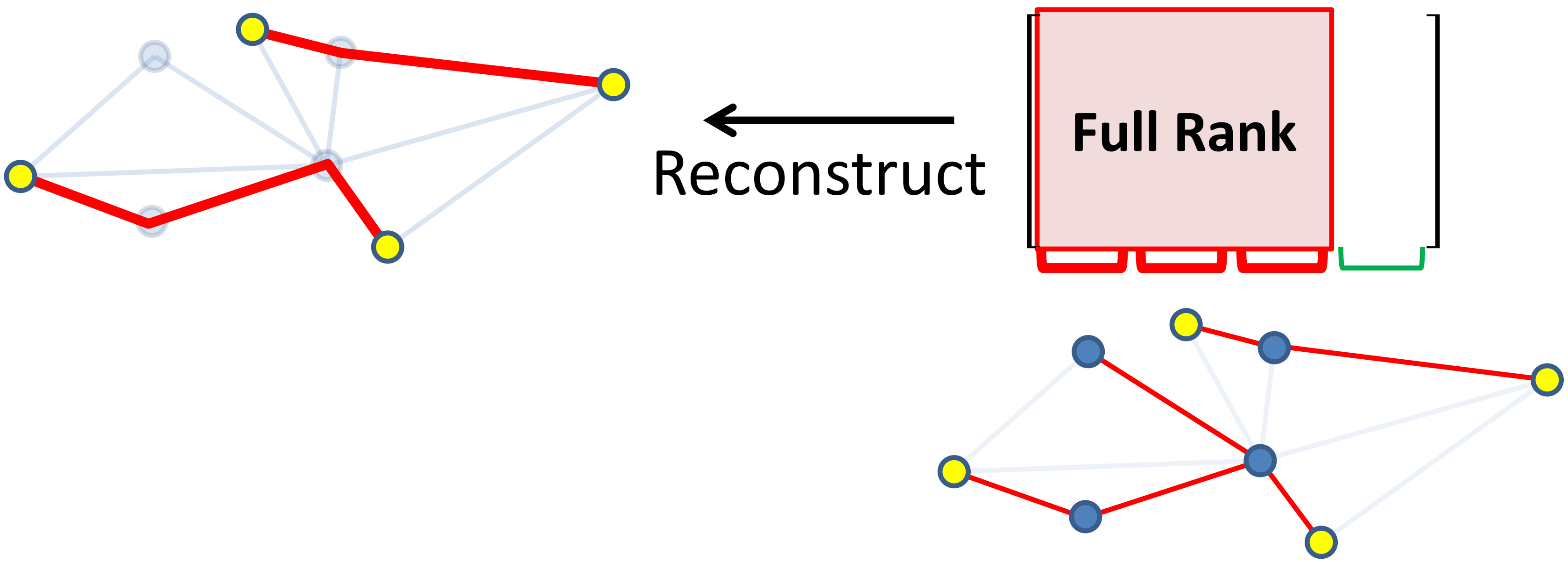
- We want a **Subgraph**





Associated Matrix

- We want a **<u>Subgraph</u>** → Edge ↔ Column-Pair



Associated Matrix

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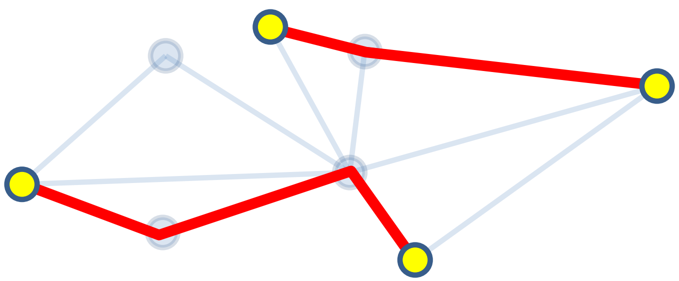
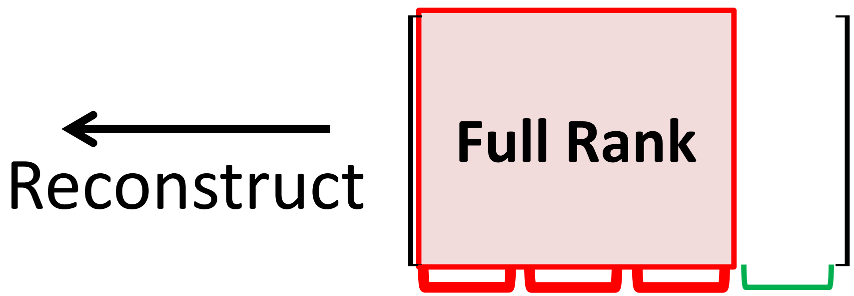
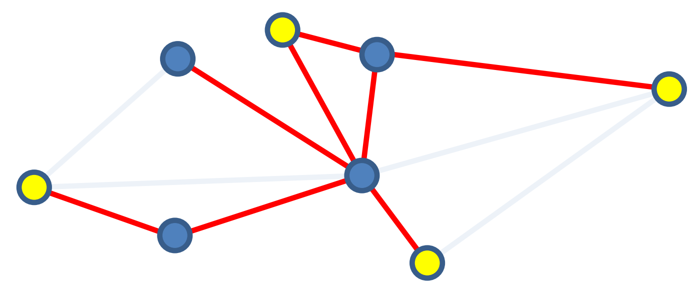
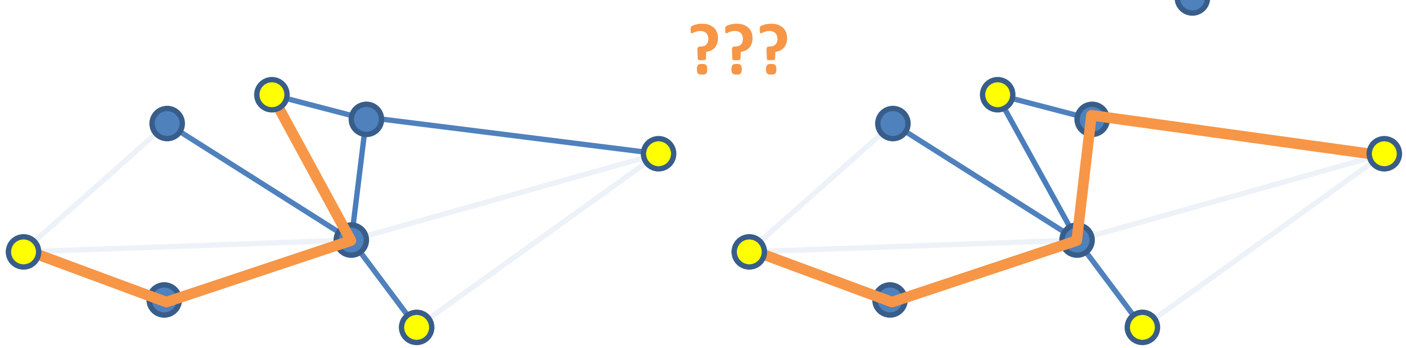




- We want <u>**Easy**</u> Reconstruction





Associated Matrix

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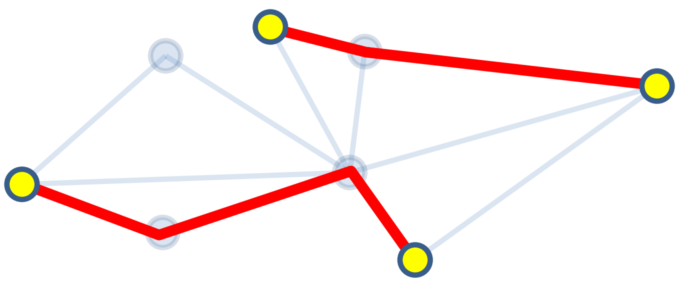

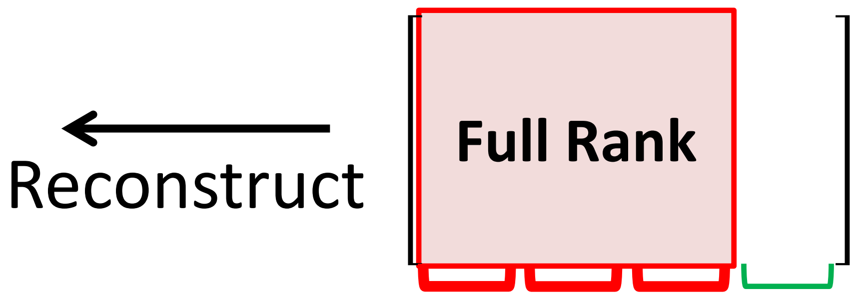

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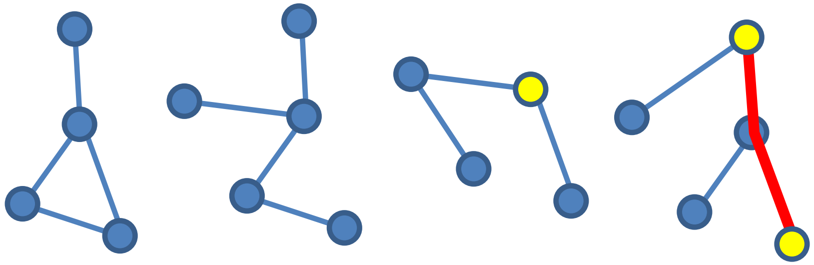


Unique and Non-zero ⟺ **Full Rank**

Sufficient Condition

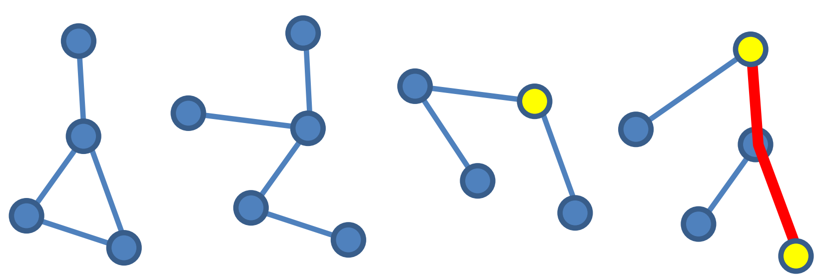
Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

$\Downarrow$

$\exists$Associated Matrix

$\mathrm{GL}(2, \mathbf{F})$: Set of All Nonsingular 2×2 Matrices over $\mathbf{F}$



Unique and Non-zero $\Leftrightarrow$ **Full Rank**

Sufficient Condition

Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

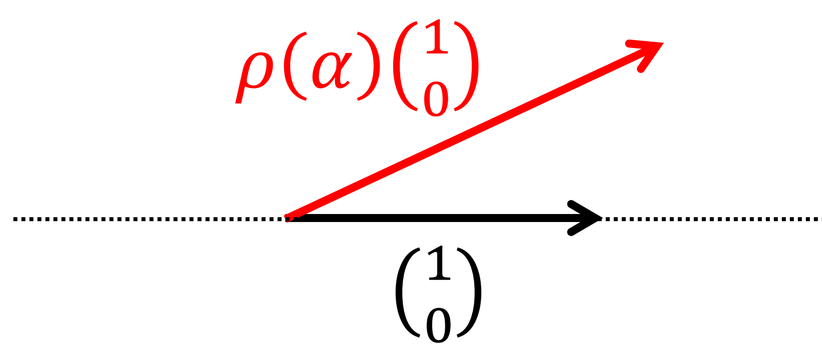
s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

$\Downarrow$

$\exists$Associated Matrix

$\mathrm{GL}(2, \mathbf{F})$: Set of All Nonsingular 2×2 Matrices over $\mathbf{F}$

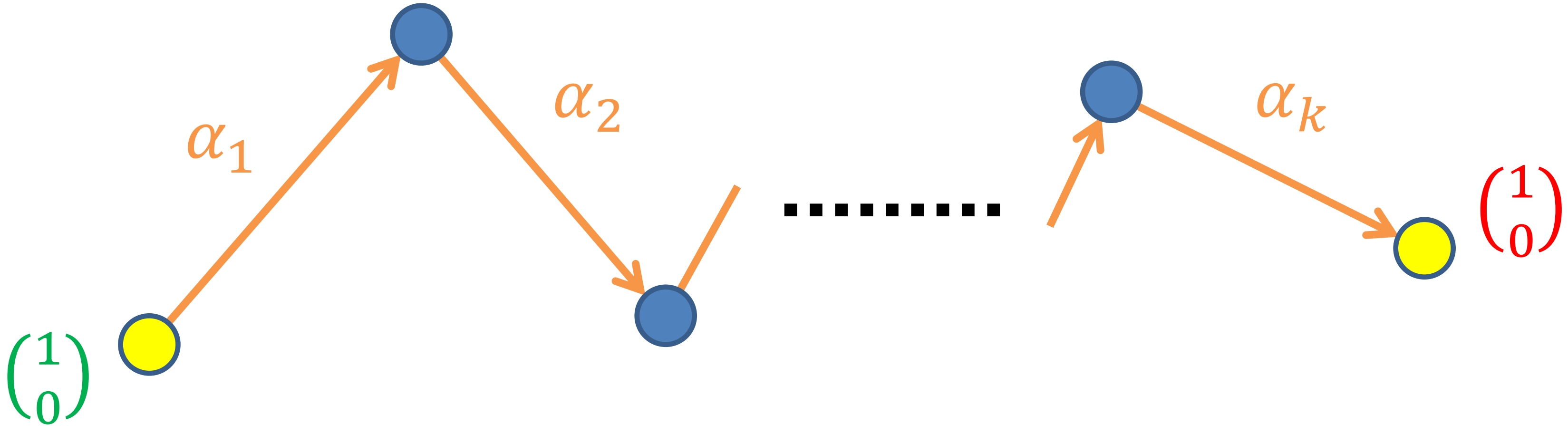
$\alpha \in \Gamma \setminus \{1_\Gamma\} \quad \Longrightarrow$



Intuitive Idea

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic
s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

- Terminals in A are associated with $\binom{1}{0}$
- Edges carry vectors with acting



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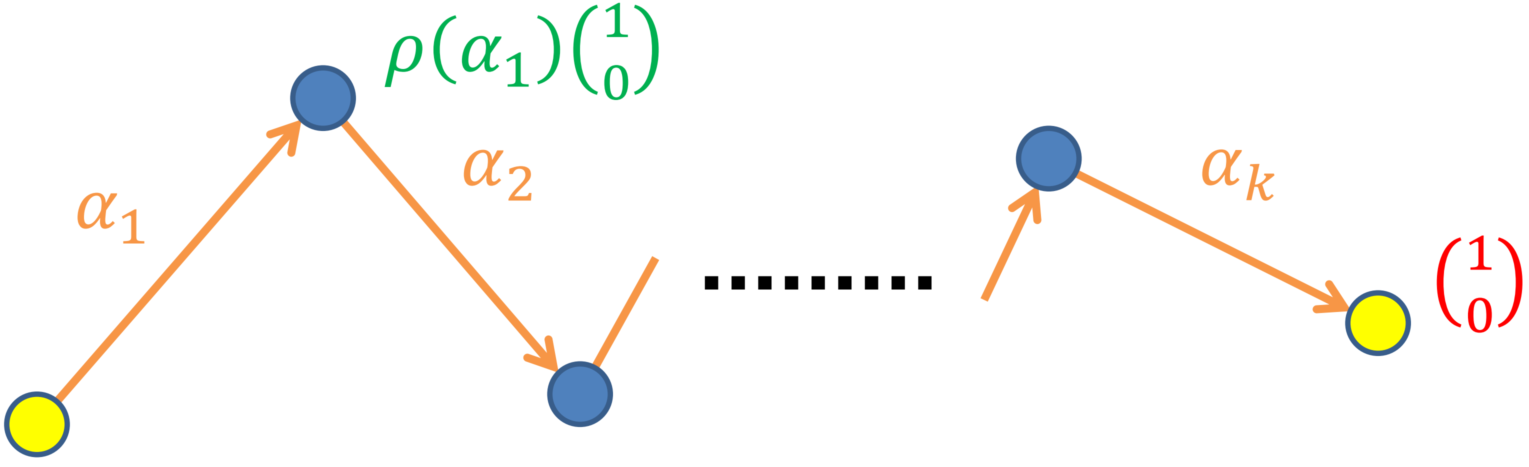

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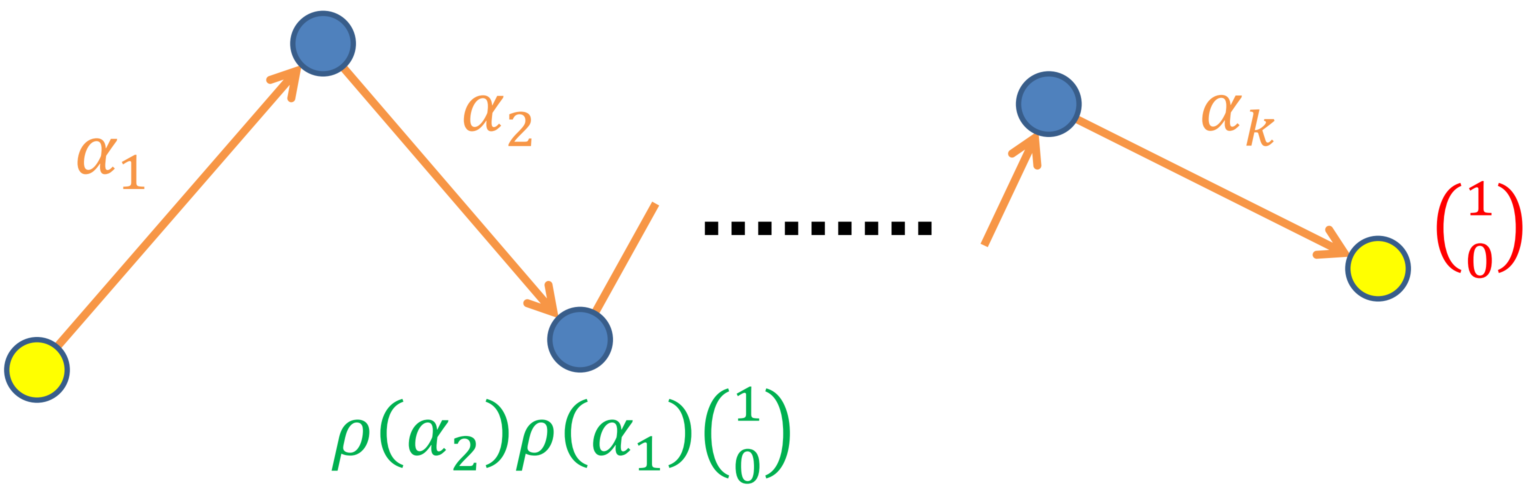

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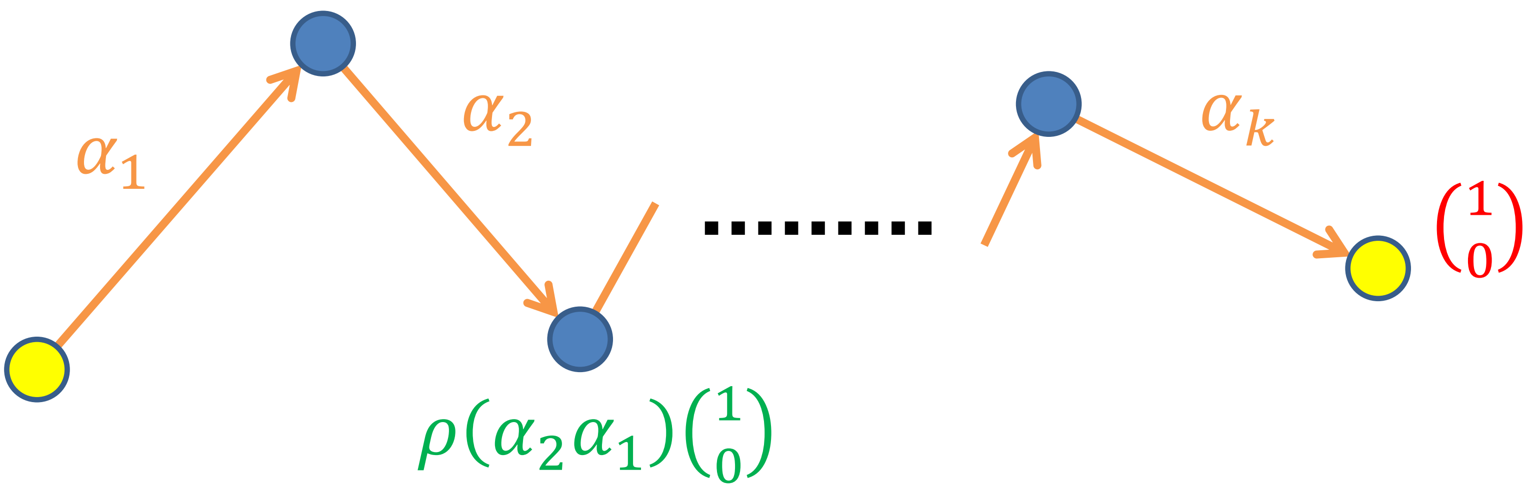

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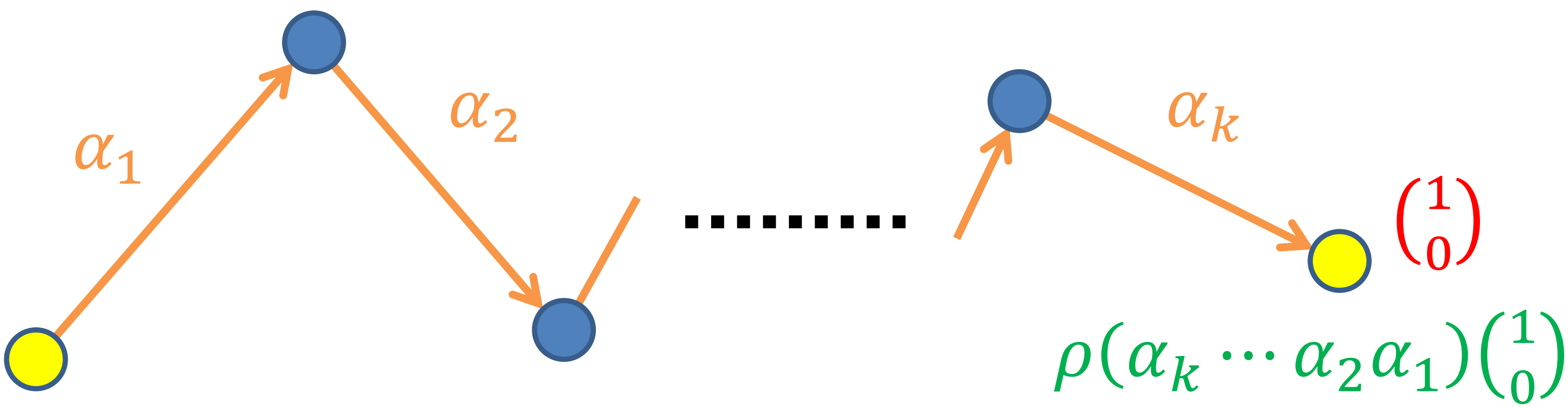

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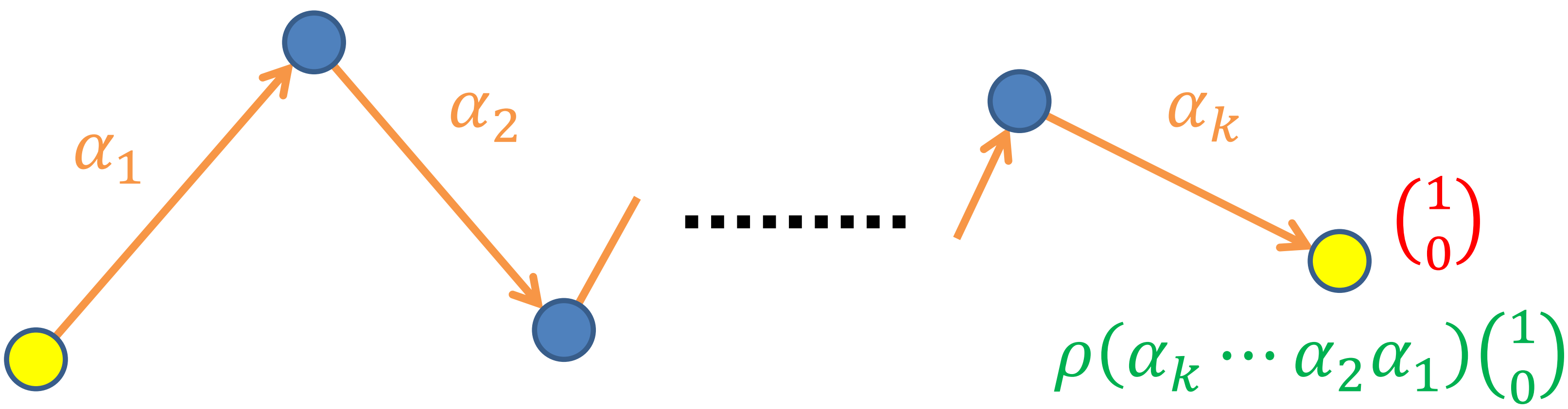
Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

$$\rho(\alpha_k \cdots \alpha_2\alpha_1)\binom{1}{0} \nparallel \binom{1}{0} \Leftrightarrow \alpha_k \cdots \alpha_2\alpha_1 \neq 1_\Gamma$$

Linearly Independent Non-zero

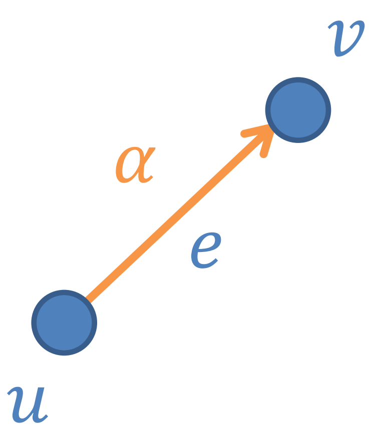


Matrix Construction (Step 1)

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

- Based on the **Incidence Matrix**
- Replace ± 1 with I_2 and $-\rho(\alpha)$



		e	
		$\mathbf{0}$	
u		1	
		$\mathbf{0}$	
v		-1	
		$\mathbf{0}$	

Matrix Construction (Step 1)

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic
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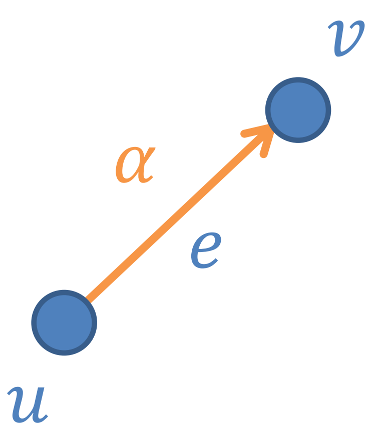


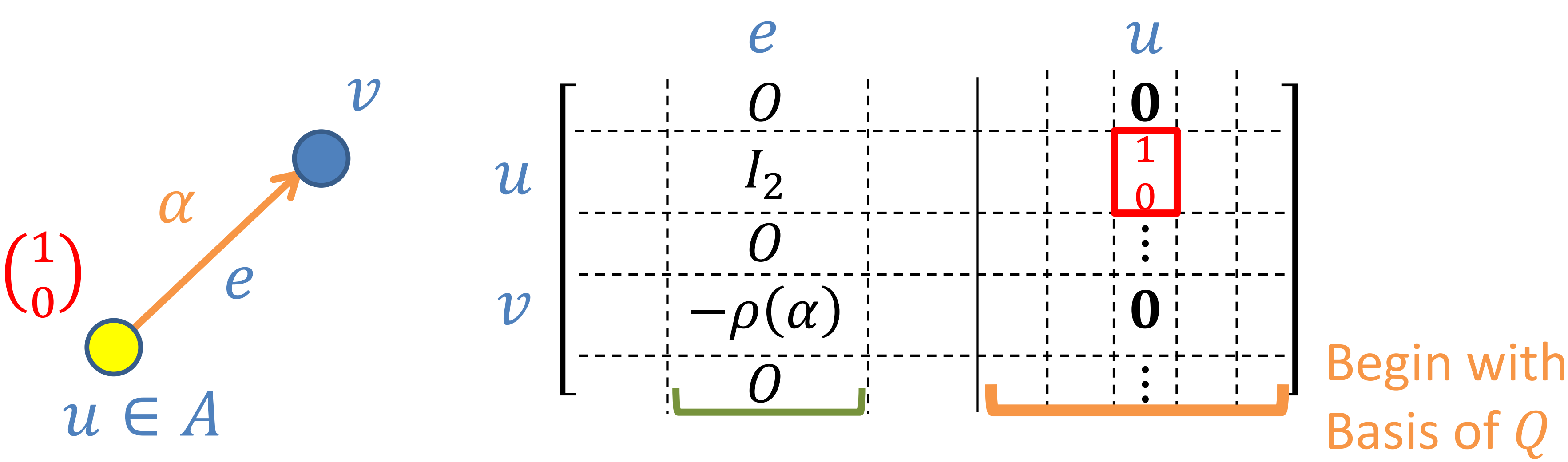
$$\begin{array}{c} \\ u \\ \\ v \\ \\ \end{array} \overset{e}{\begin{bmatrix} \mathbf{0} \\ 1 \\ \mathbf{0} \\ -1 \\ \mathbf{0} \end{bmatrix}} \to \begin{array}{c} \\ u \\ \\ v \\ \\ \end{array} \overset{e}{\begin{bmatrix} O \\ I_2 \\ O \\ -\rho(\alpha) \\ O \end{bmatrix}}$$

Paired

Matrix Construction (Step 2)

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic
s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

- $Q := \left\{ x \in (\mathbf{F}^2)^V \mid x_u \parallel \binom{1}{0} \ (u \in A),\ x_v = \mathbf{0} \ (v \notin A) \right\}$
- **Linear Independence** in $(\mathbf{F}^2)^V / Q$



Matrix Construction (Step 2)

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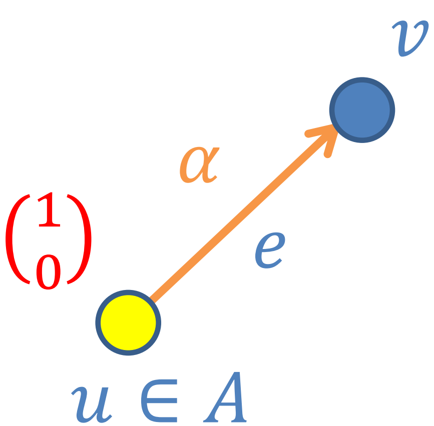
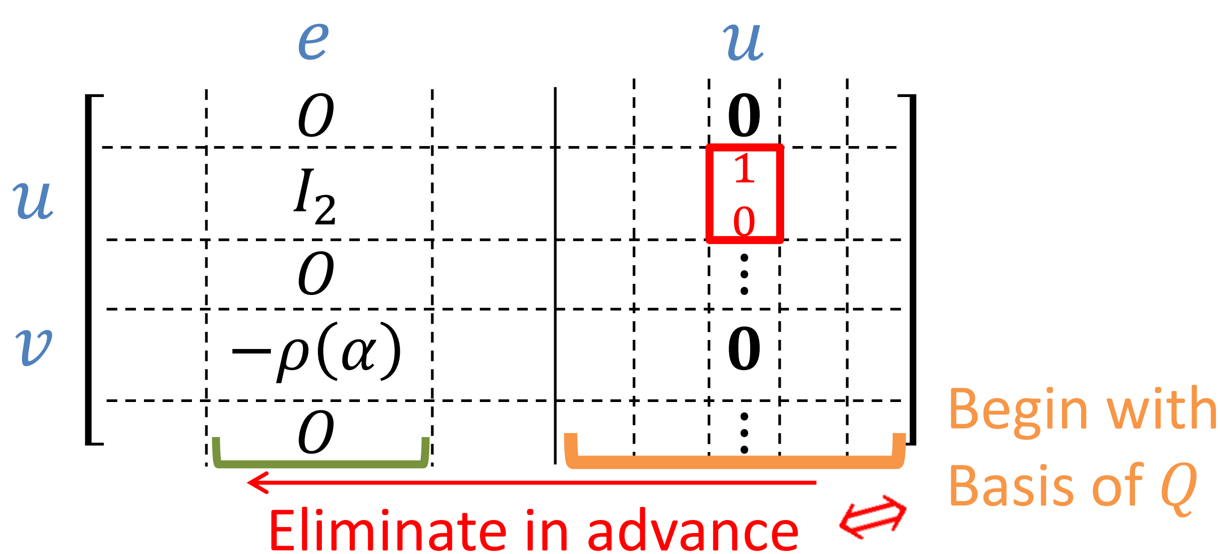




Ex. 1 Mader's $\mathcal{S}$-paths

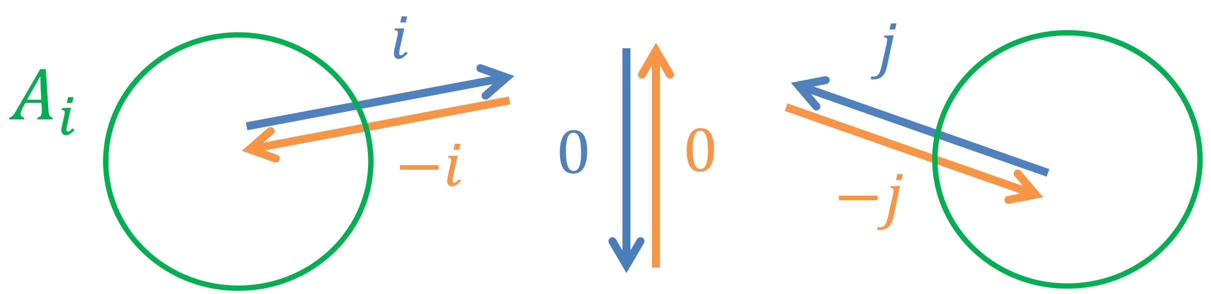
Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \iff \alpha = 1_\Gamma$

$\Gamma = \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$, $\mathbf{F} = \mathbf{Q}$: Rational Field

$$\rho(k) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \qquad (k \in \mathbf{Z})$$

$$\rho(k)\binom{1}{0} = \binom{1}{k} \parallel \binom{1}{0} \iff k = 0$$



$\mathcal{S}$-paths
⇕
Non-zero

Ex. 1 Mader's $\mathcal{S}$-paths

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic
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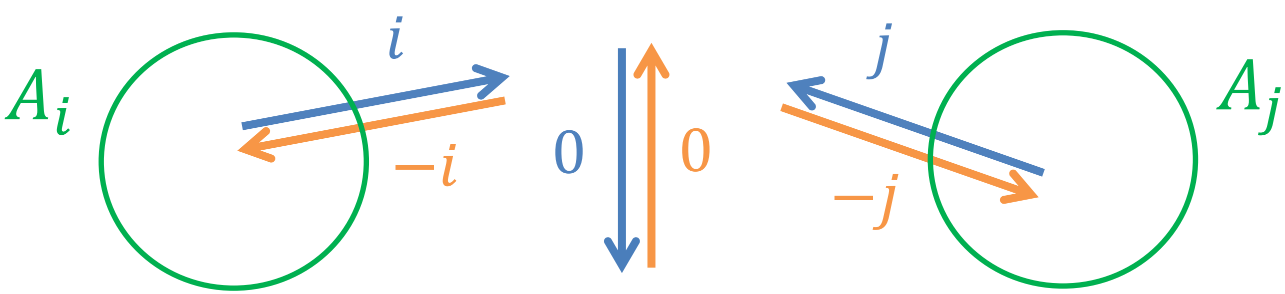


$\mathcal{S}$-paths
⇕
Non-zero

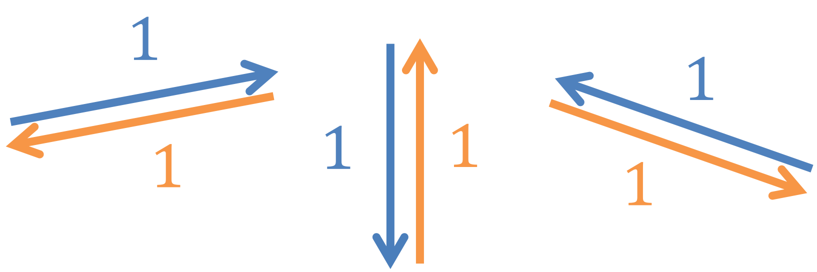
Ex. 2 Odd-Length A-paths

Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic
s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \iff \alpha = 1_\Gamma$

$\Gamma = \mathbf{Z}_2 = \{0, 1\} \pmod 2$, $\mathbf{F}$: Arbitrary Field

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho(0)\binom{1}{0} = \binom{1}{0} \parallel \binom{1}{0}, \quad \rho(1)\binom{1}{0} = \binom{0}{1} \nparallel \binom{1}{0}$$



Odd-Length
$\Updownarrow$
Non-zero

Ex. 2 Odd-Length A-paths

Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

$\Gamma = \mathbf{Z}_2 = \{0, 1\}\ (\mathrm{mod}\ 2)$, $\mathbf{F}$: Arbitrary Field

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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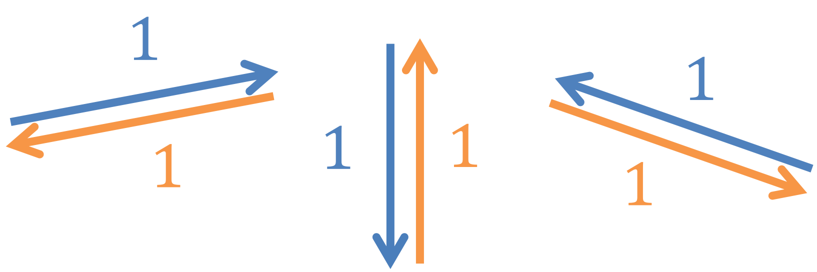


Odd-Length

⇕

Non-zero

Sufficient Condition

Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{GL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \iff \alpha = 1_\Gamma$

$\Downarrow$

Reducible to Linear Matroid Parity

$\mathrm{GL}(2, \mathbf{F})$: Set of All Nonsingular 2×2 Matrices over $\mathbf{F}$

Sufficient Condition

Γ: Group, $\mathbf{F}$: Field, $\rho\colon \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ Homomorphic

s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

$\Downarrow$

Reducible to Linear Matroid Parity

$\mathrm{GL}(2, \mathbf{F})$: Set of All Nonsingular 2×2 Matrices over $\mathbf{F}$

$$\mathrm{PGL}(2, \mathbf{F}) \coloneqq \mathrm{GL}(2, \mathbf{F}) / \left\{ \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \mid k \in \mathbf{F} \setminus \{0\} \right\}$$

$$\forall Z \in \mathrm{GL}(2, \mathbf{F}), \forall k \in \mathbf{F} \setminus \{0\}, \quad Z \sim kZ$$

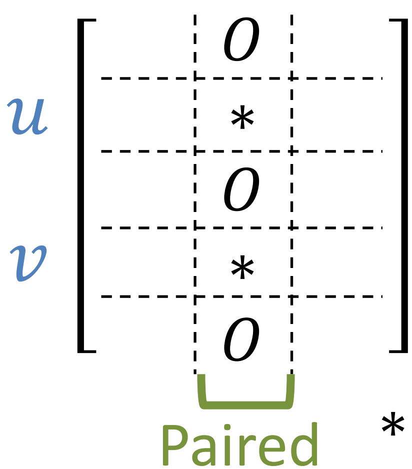
Necessary and Sufficient Condition

Γ: Group, $\mathbf{F}$: Field, $\rho: \Gamma \to \mathrm{PGL}(2, \mathbf{F})$ Homomorphic
s.t. $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow \alpha = 1_\Gamma$

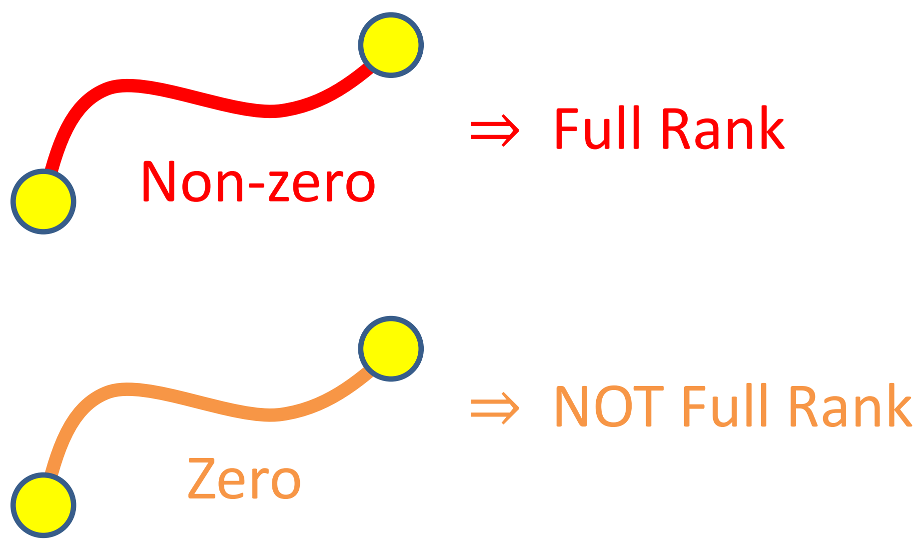
$$\Updownarrow$$

Reducible to L.M.P. with **Coherent Representation**

$e = uv \in E$



$*$: 2×2 Matrix



Conclusion

- **Packing Non-zero A-paths** is efficiently solvable via **Linear Matroid Parity** under some **Group Representability** condition.

$O(|V|^5)$-time → $O(|V|^{2.373})$-time

[Chudnovsky, Cunningham, Geelen 2008] [Cheung, Law, Leung 2014]

- The same condition is also **Necessary** for **Reasonable Reduction** to **L.M.P.**

Extension

Our Result is **Extendable** to the following cases

- **Subgroup-Forbidden** Model

  Γ': Proper Subgroup of Γ, Set of **Forbidden** Labels

  Idea $\rho(\alpha)\binom{1}{0} \parallel \binom{1}{0} \Leftrightarrow$ ~~$\alpha = 1_\Gamma$~~ $\alpha \in \Gamma'$

- **Weighted** Setting $\rightarrow$ Weighted Linear Matroid Parity

  [Iwata 2013] [Pap 2013]

  $c: E \rightarrow \mathbf{R}$ (**Edge Cost**), $k \in \mathbf{Z}_+$

  Finding **Minimum Cost** k Disjoint Non-zero A-paths

  Idea Add Dummy Terminals (cf. Weighted Matching)