Realizing Symmetric Set Functions as Hypergraph Cut Capacity

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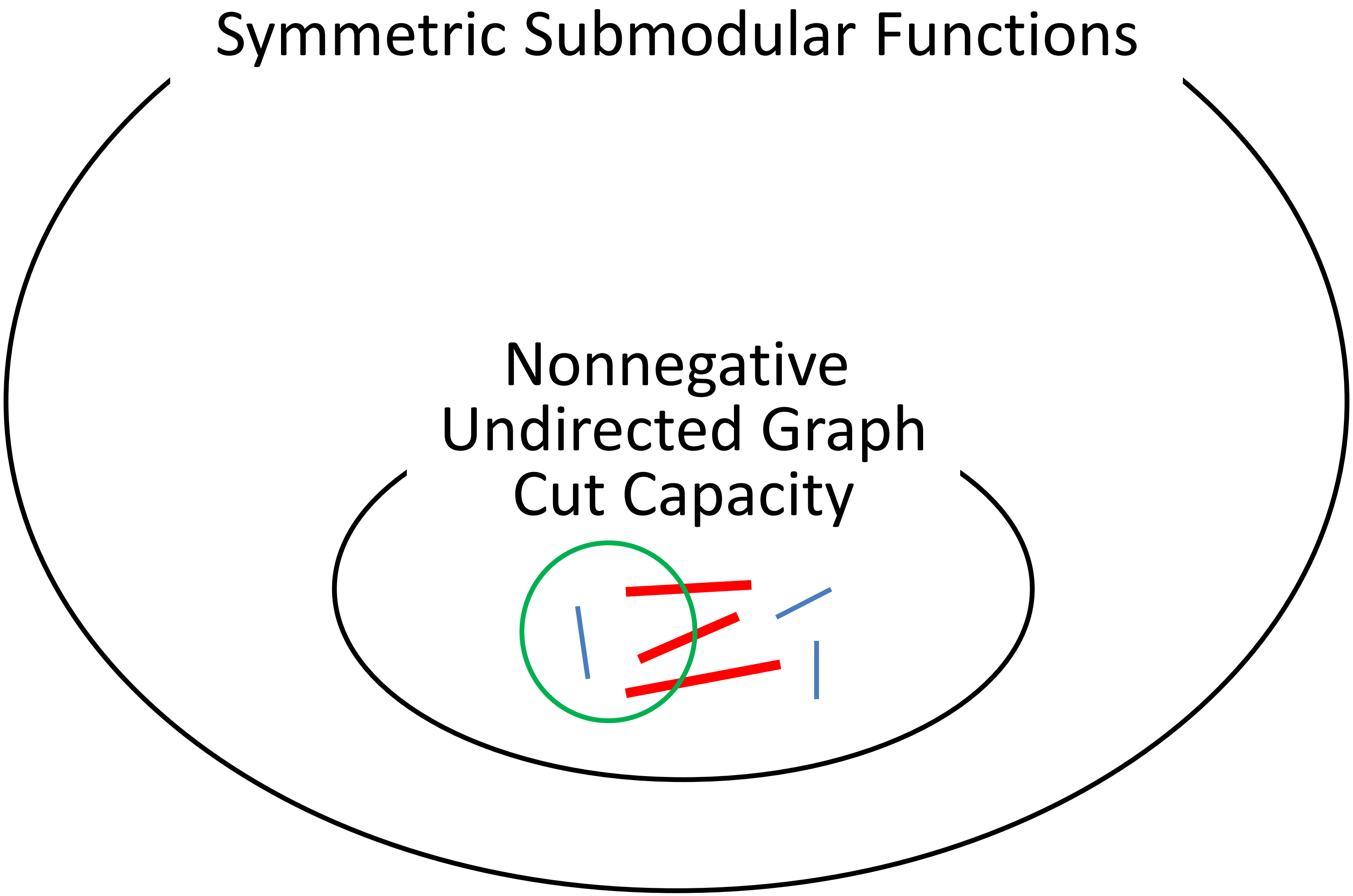
Background


Symmetric Submodular Functions
Nonnegative
Undirected Graph
Cut Capacity

Background

$f(X) = f(V \setminus X)$

Symmetric **Submodular** Functions

$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$

Nonnegative

Undirected Graph

Cut Capacity

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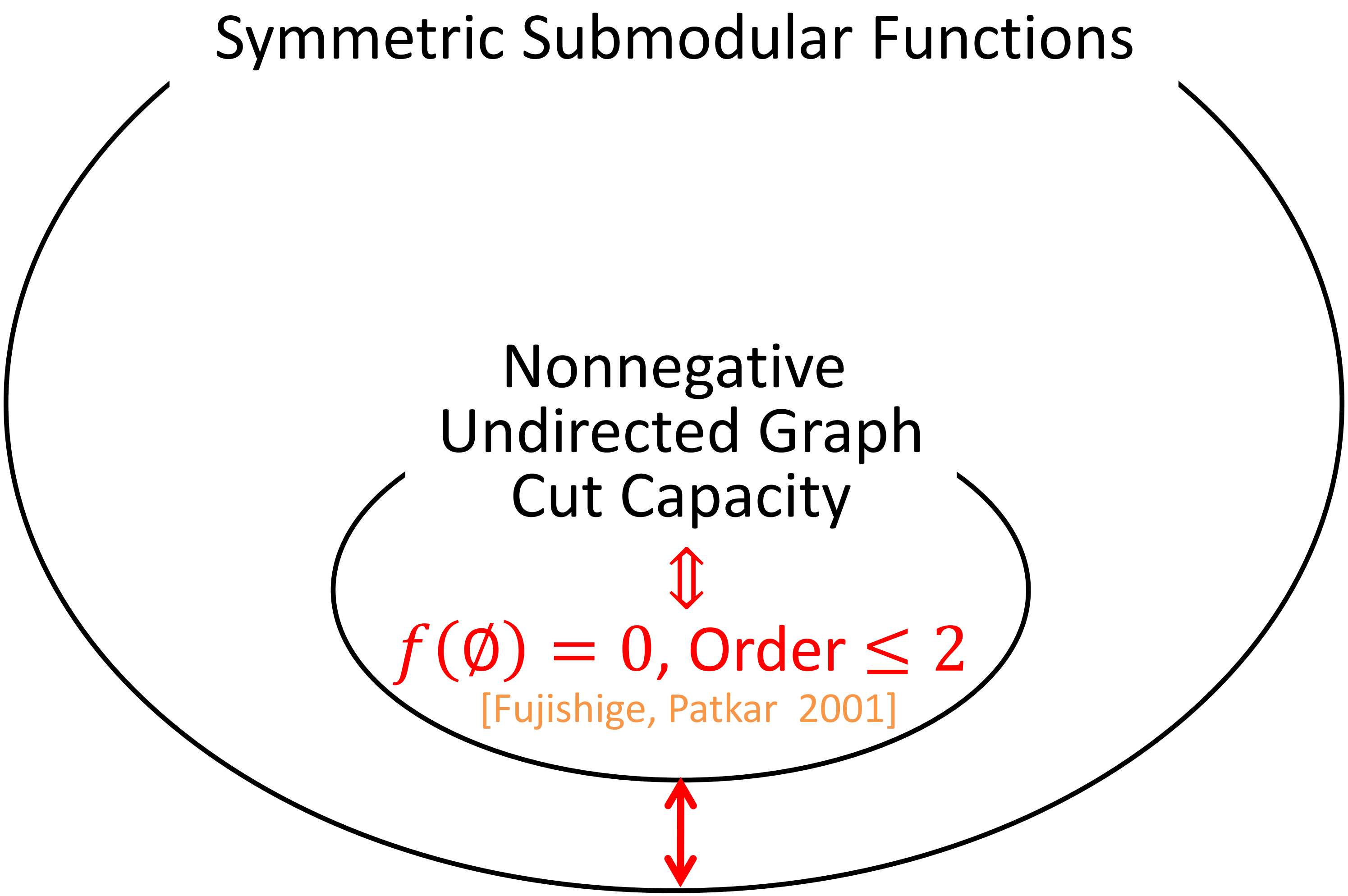

Symmetric Submodular Functions
Nonnegative
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⇕
$f(\emptyset) = 0$, Order ≤ 2
[Fujishige, Patkar 2001]

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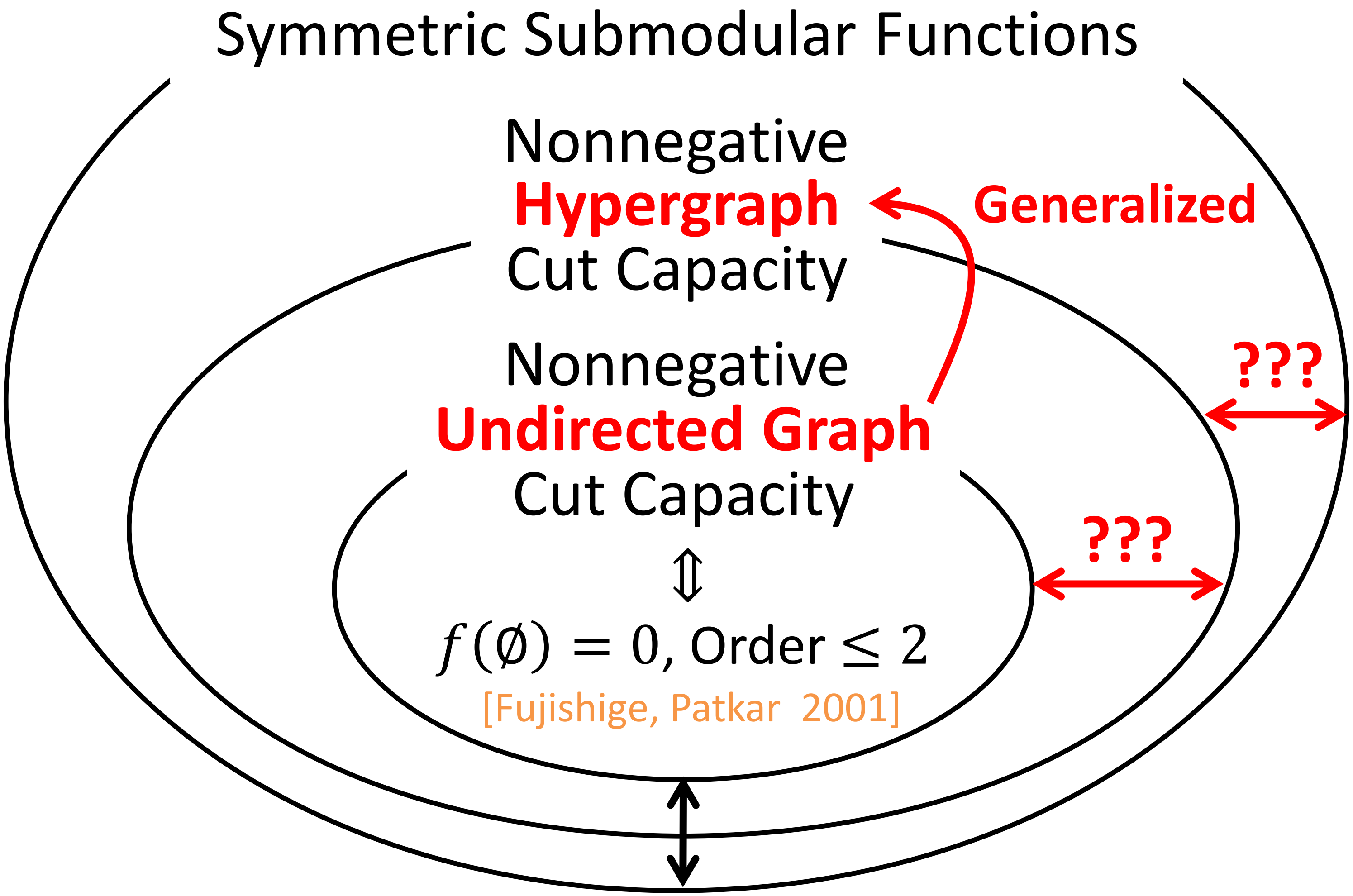

Symmetric Submodular Functions
Nonnegative
Hypergraph
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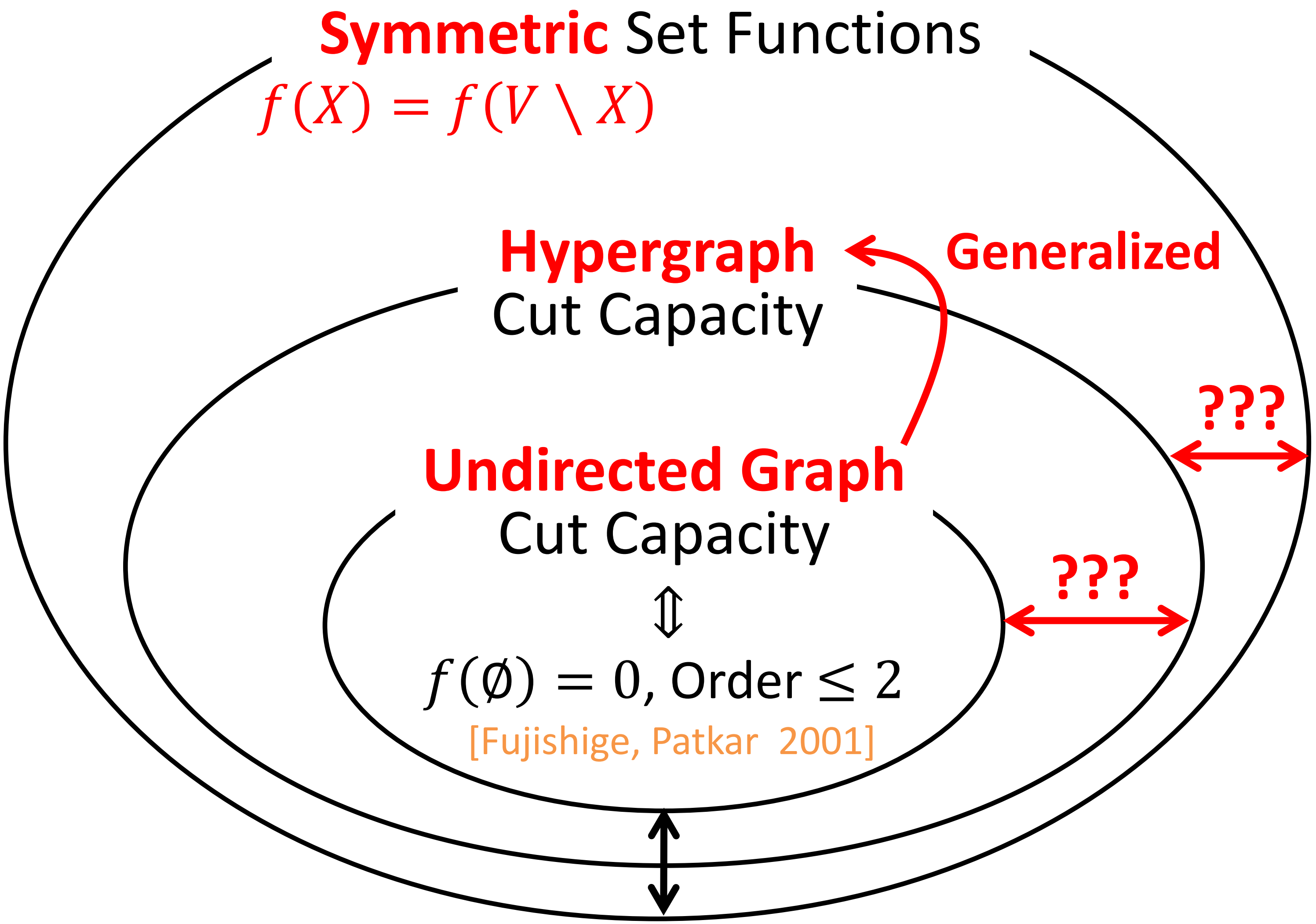
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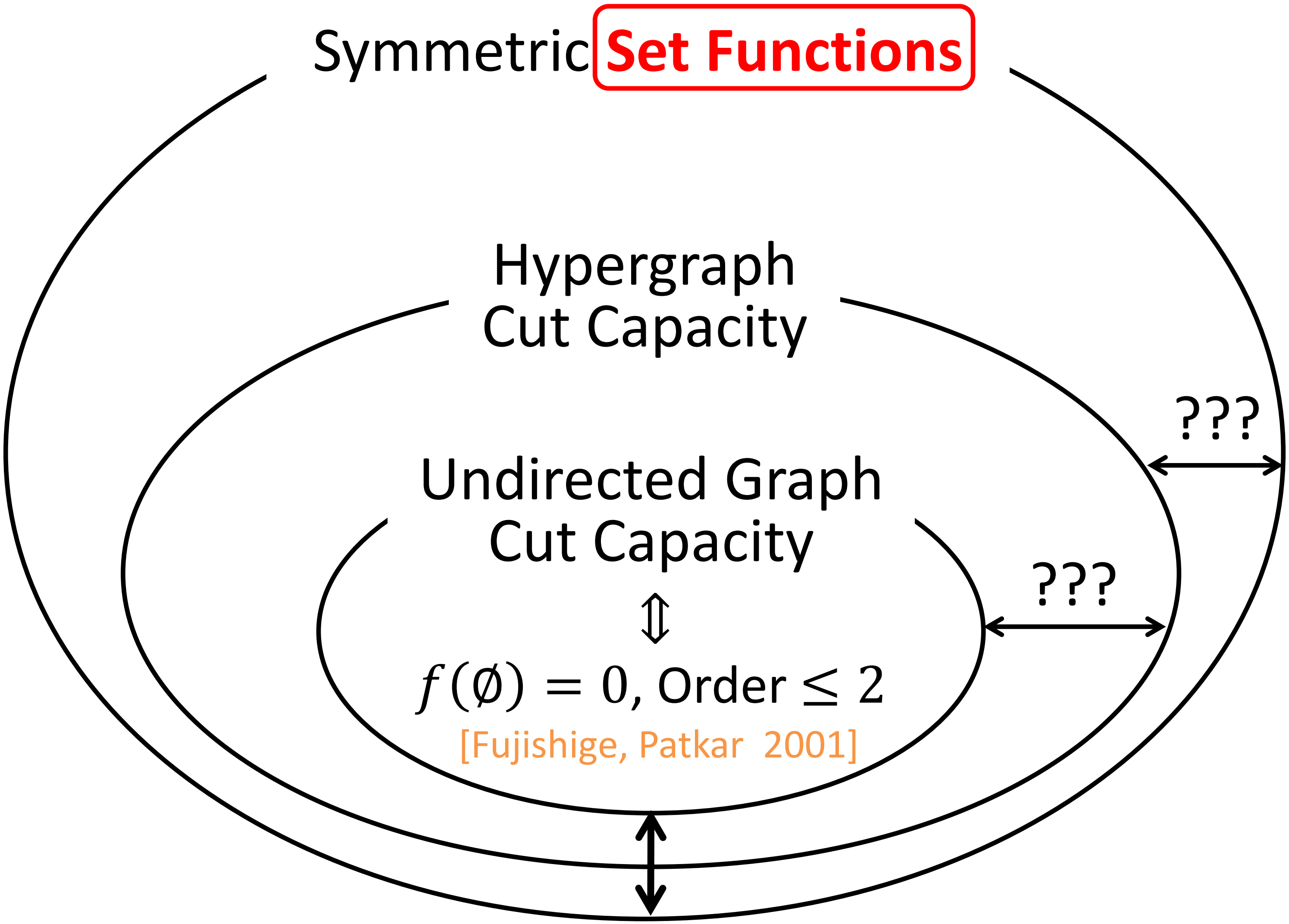

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Set Functions

||

Functions on set families

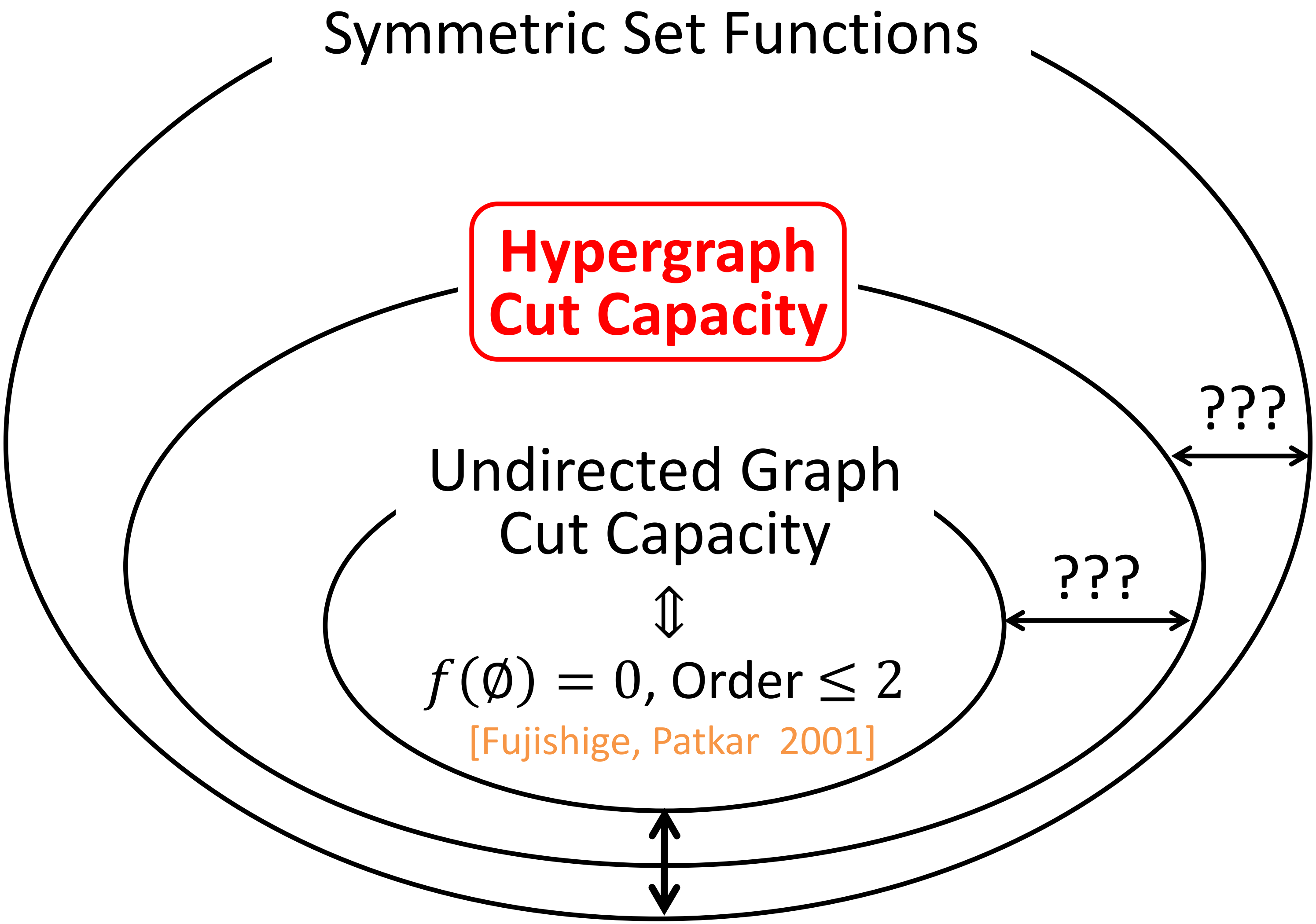
- V: finite set (**ground set**)
- $\mathcal{D} \subseteq 2^V = \{ X \mid X \subseteq V \}$: family of subsets (**domain**)
- R: set of values (**codomain**)

$f: \mathcal{D} \to R$ is called a **set function**.

* We assume $R = \mathbf{R} = \{ r \mid r\text{: real} \}$.

* We assume $\mathcal{D} = 2^V$, unless any notice.

Background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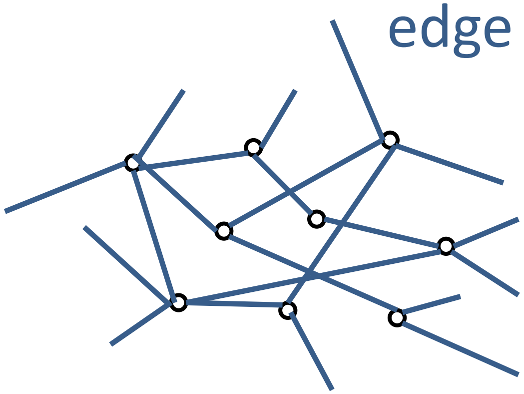
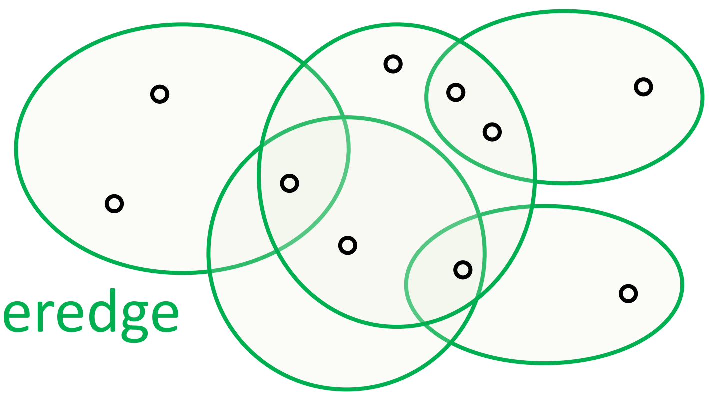
Hypergraphs

Undirected graph —Generalized→ **Hypergraph**





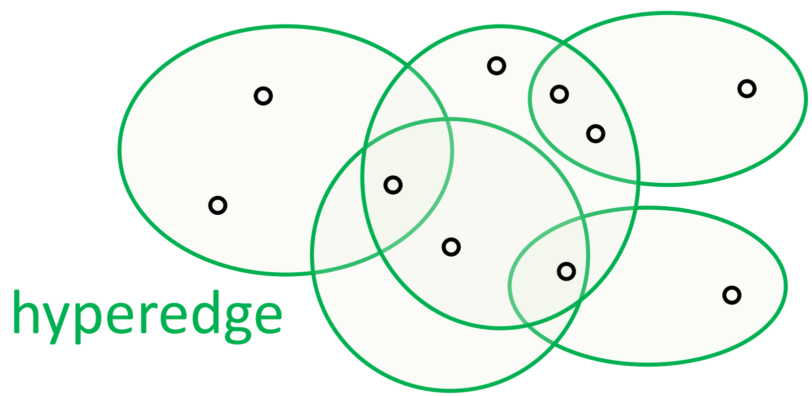
Each edge connects **two vertices**.

Each hyperedge connects **an arbitrary number of vertices**.

Hypergraphs

- V: finite set (**vertex set**)
- $\mathcal{E} \subseteq 2^V$: family of subsets (**hyperedge set**)

$\mathcal{H} = (V, \mathcal{E})$ is called a **hypergraph**.



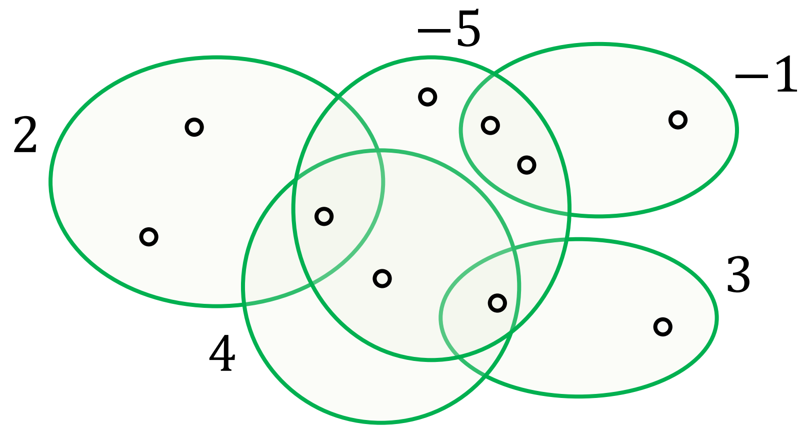
Hypernetworks

- V: finite set (**vertex set**)
- $\mathcal{E} \subseteq 2^V$: family of subsets (**hyperedge set**)

$\mathcal{H} = (V, \mathcal{E})$ is called a **hypergraph**.

- $c: \mathcal{E} \to \mathbf{R}$, real-valued set function (**capacity function**)

$\mathcal{N} = (\mathcal{H}, c)$ is called a **hypernetwork**.



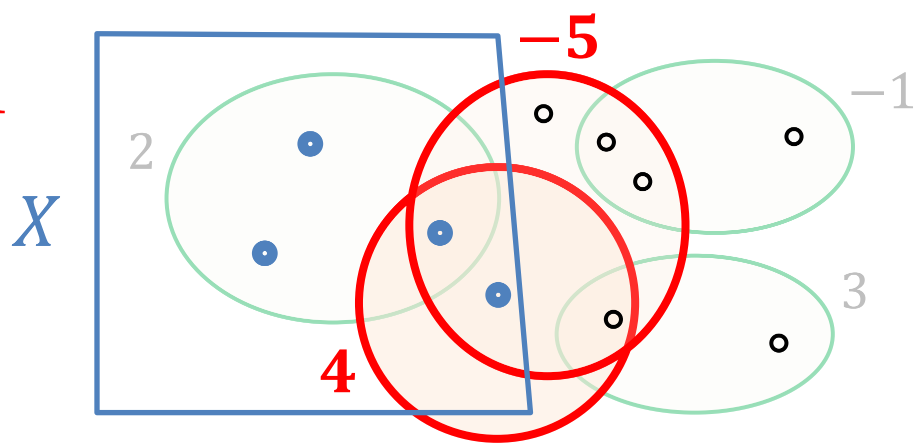
Cut Capacity

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

The **cut capacity function** $\kappa_{\mathcal{N}}: 2^V \to \mathbf{R}$

$$\kappa_{\mathcal{N}}(X) := \sum_{E \in \mathcal{E}} \{ c(E) \mid E \cap X \neq \emptyset \neq E \setminus X \}$$

$\kappa_{\mathcal{N}}(X) = -1$



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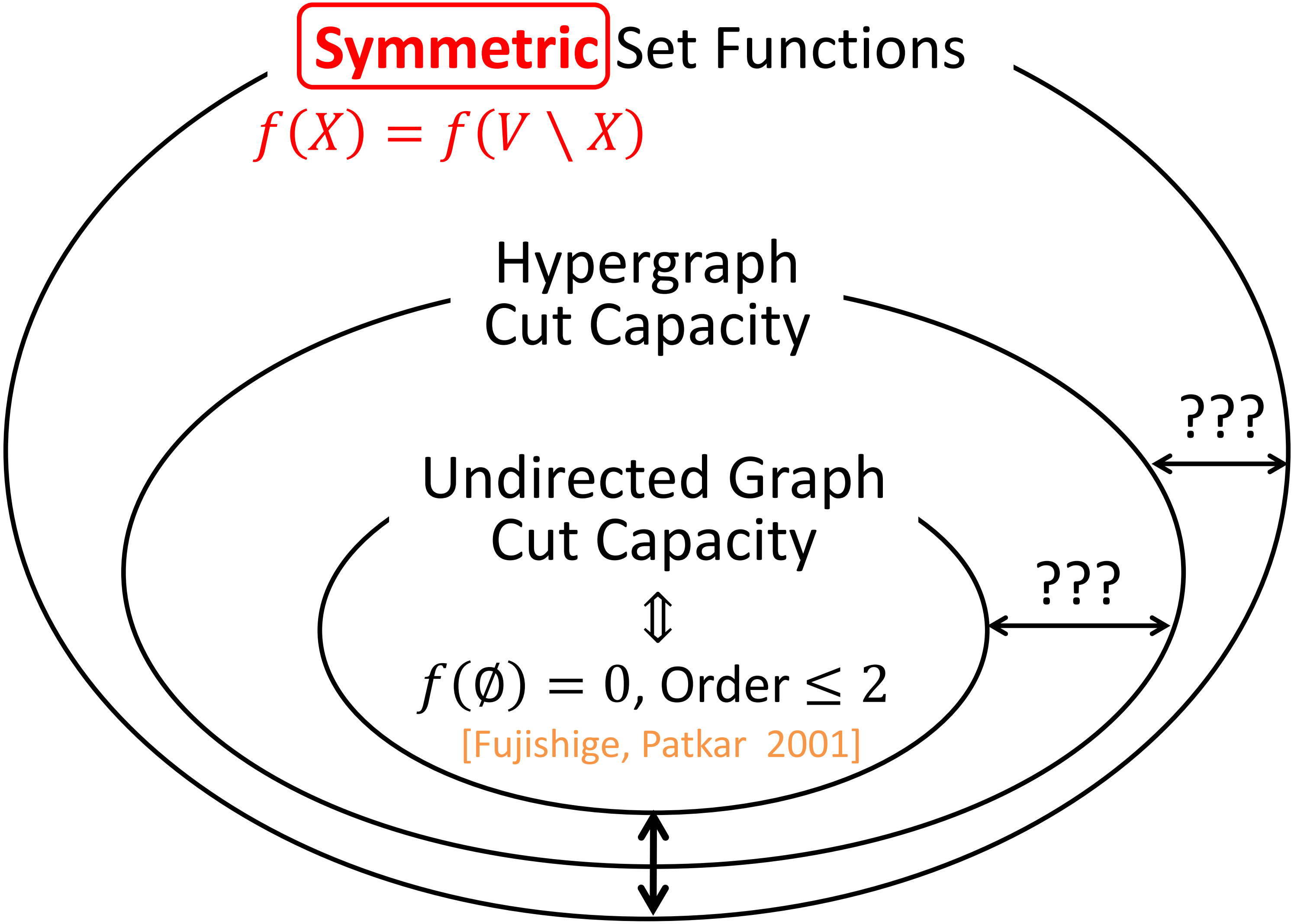
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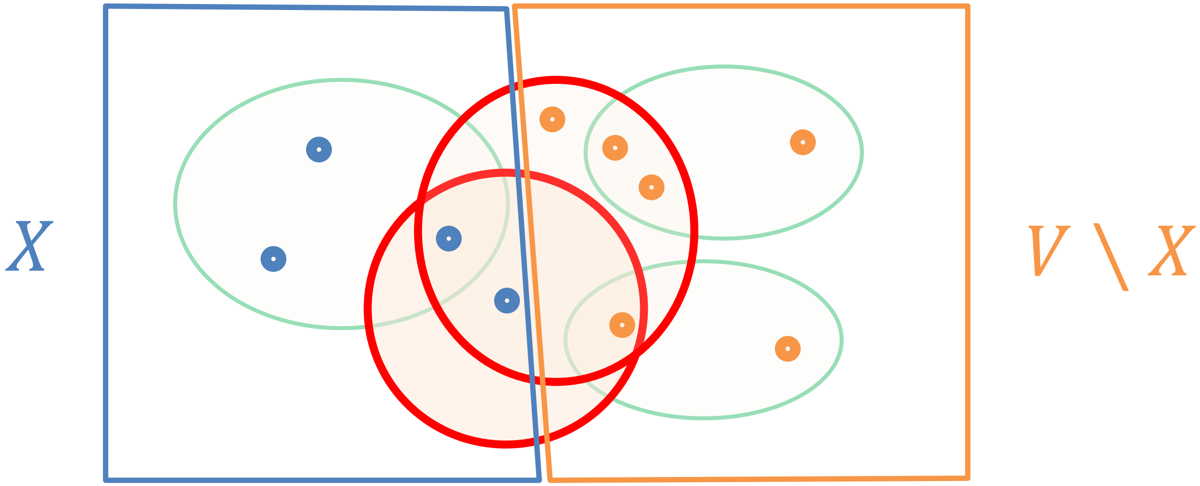
Symmetry of Cut Capacity

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$\kappa_{\mathcal{N}}$ is **symmetric**, *i.e.*,

$$\kappa_{\mathcal{N}}(X) = \kappa_{\mathcal{N}}(V \setminus X) \quad (X \subseteq V)$$



Today's Talk

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

The cut capacity function $\kappa_{\mathcal{N}}: 2^V \to \mathbf{R}$ is a set function.

- Which set functions can be realized as cut capacity?

- When possible, how can we realize them?

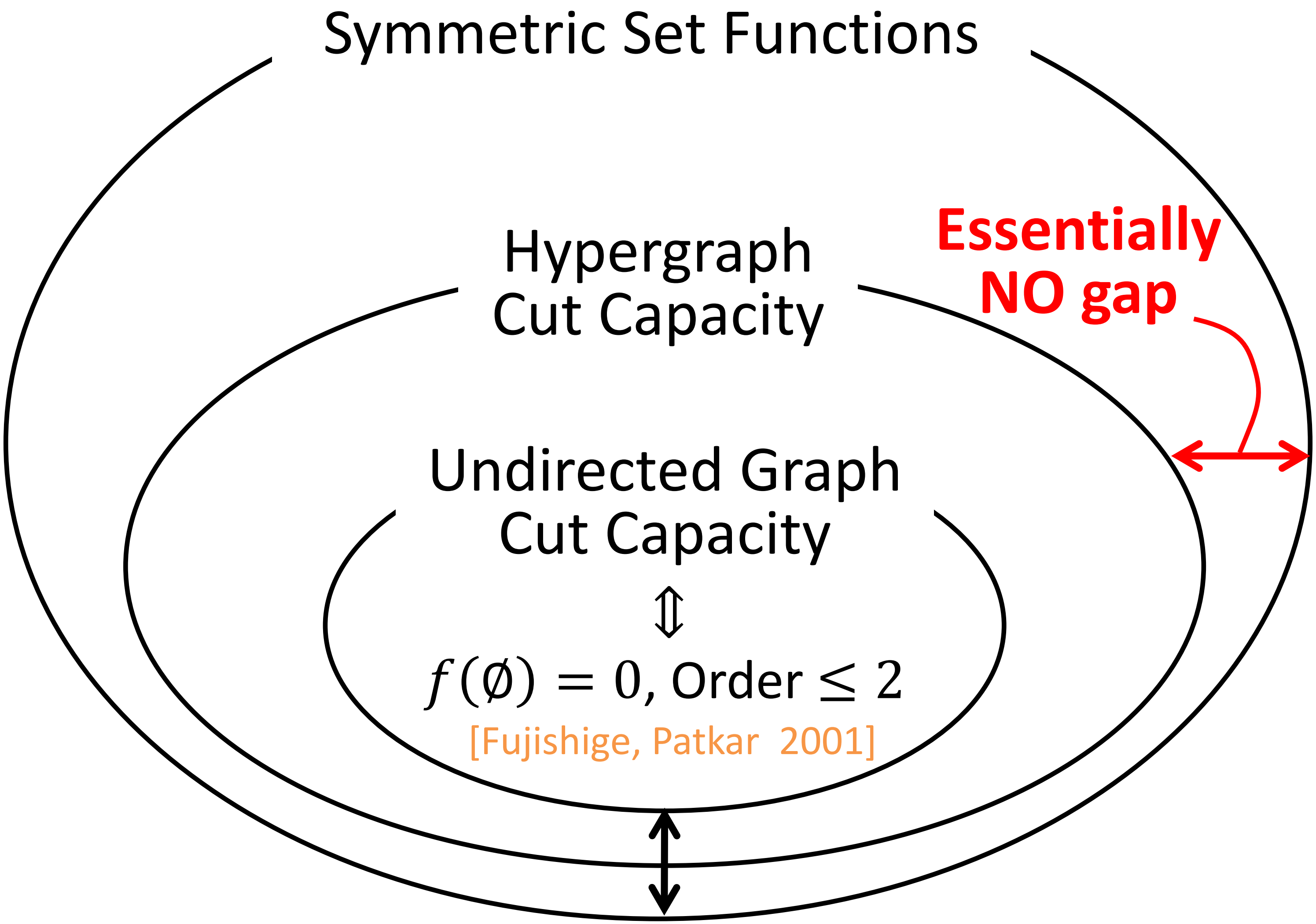
Today's Talk

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

The cut capacity function $\kappa_{\mathcal{N}}: 2^V \to \mathbf{R}$ is a set function.

- Which set functions can be realized as cut capacity?
  - Essentially **ALL** **symmetric** set functions!
  - **Submodularity** is **far from sufficient** when $c \geq 0$.
- When possible, how can we realize them?
  - We give several **standard forms** of hypergraphs by **restricting available hyperedges**.

Overview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Realizability by Hypergraph

$f: 2^V \to \mathbf{R}$, symmetric

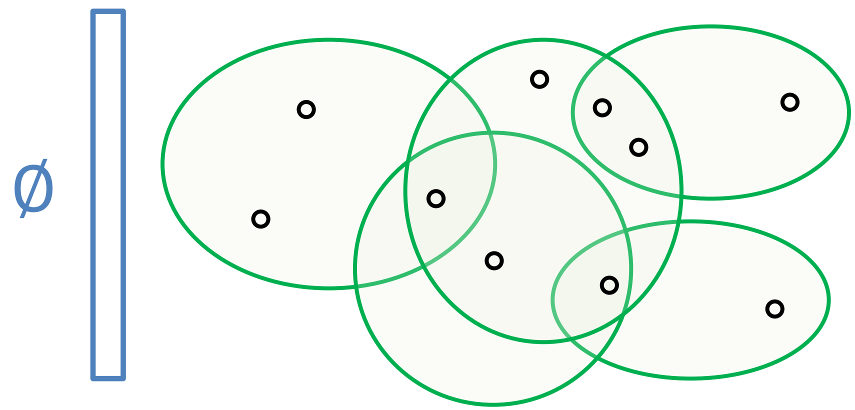
f is **realizable** as the cut cap. func. of a **hypernetwork**.

$$\Updownarrow$$

$$f(\emptyset) = 0.$$

[Y. 2015]?

$\kappa_{\mathcal{N}}(\emptyset) = 0$



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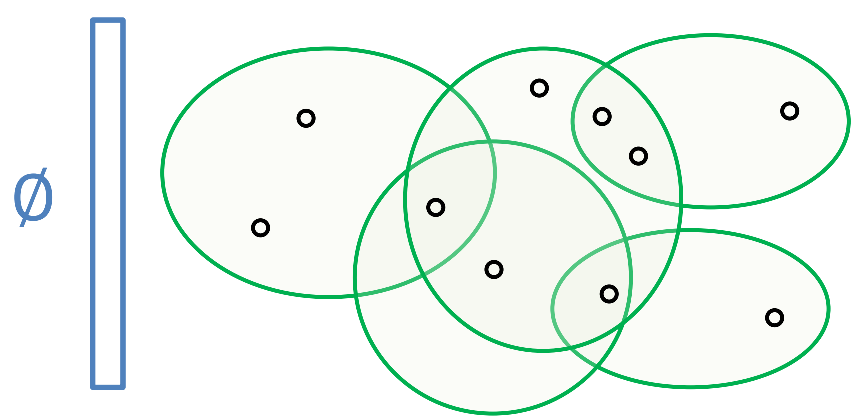
Corollary of [Grishuhin 1989] ~~[Y. 2015]?~~

$\kappa_{\mathcal{N}}(\emptyset) = 0$



Realizability by Hypergraph

$f : 2^V \to \mathbf{R}$, symmetric

f is **realizable** as the cut cap. func. of a **hypernetwork** with hyperedges of **size at most k**.

$$\Updownarrow$$

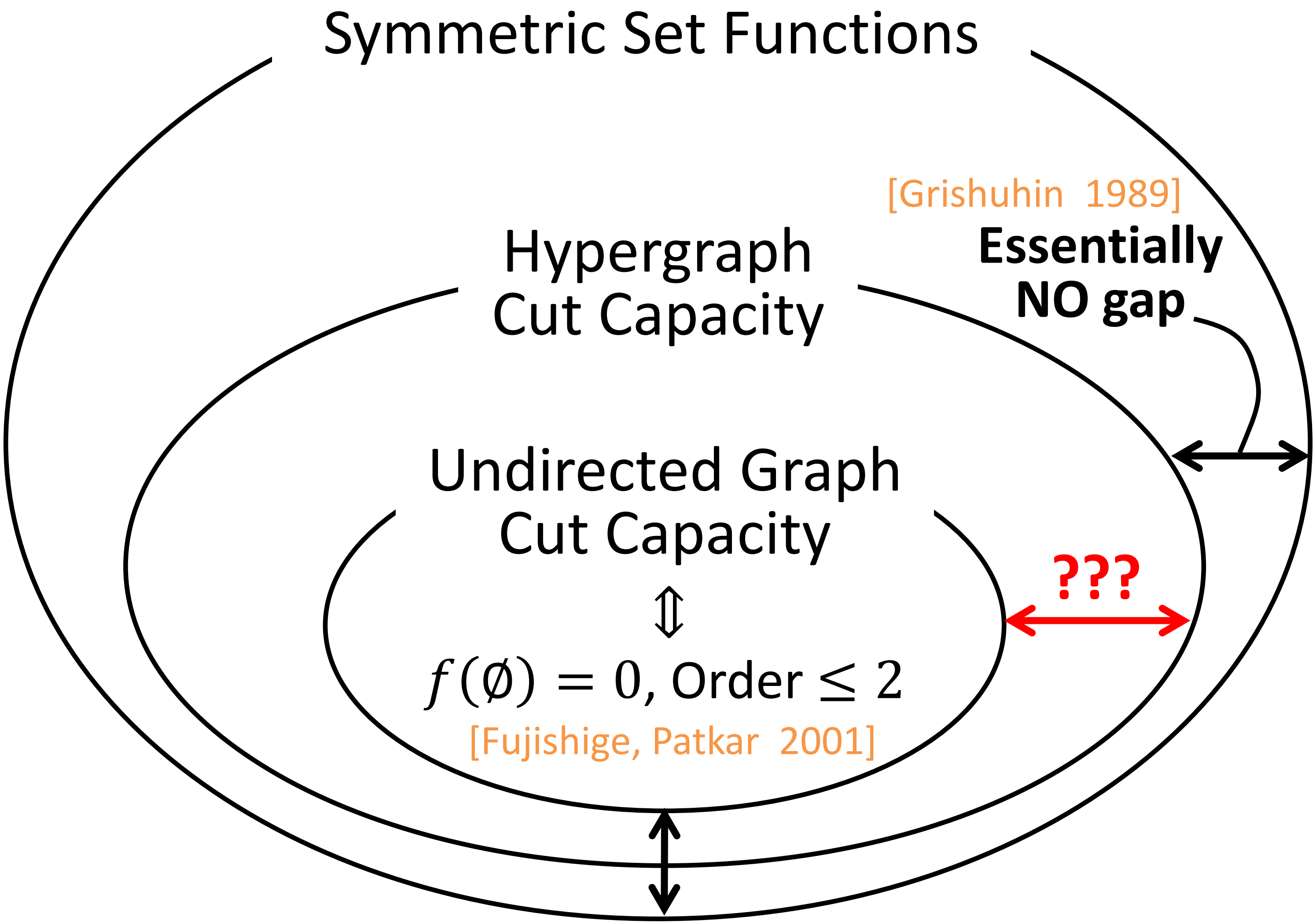
$$f(\emptyset) = 0,$$

f is of **order at most k**.

[Y. 2015]

Extends the case of undirected graphs [Fujishige, Patkar 2001].
(*i.e.*, $k = 2$)

Overview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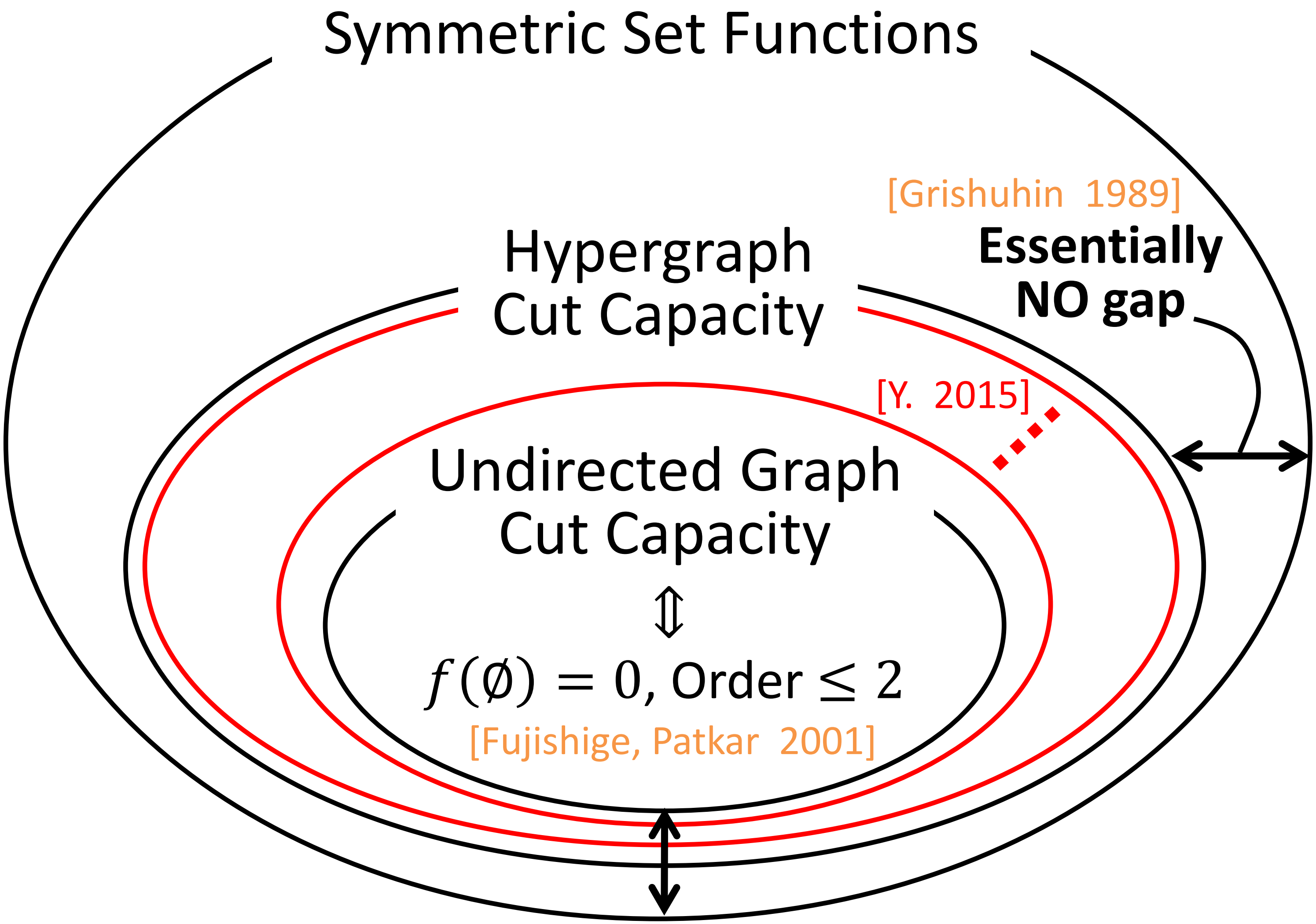

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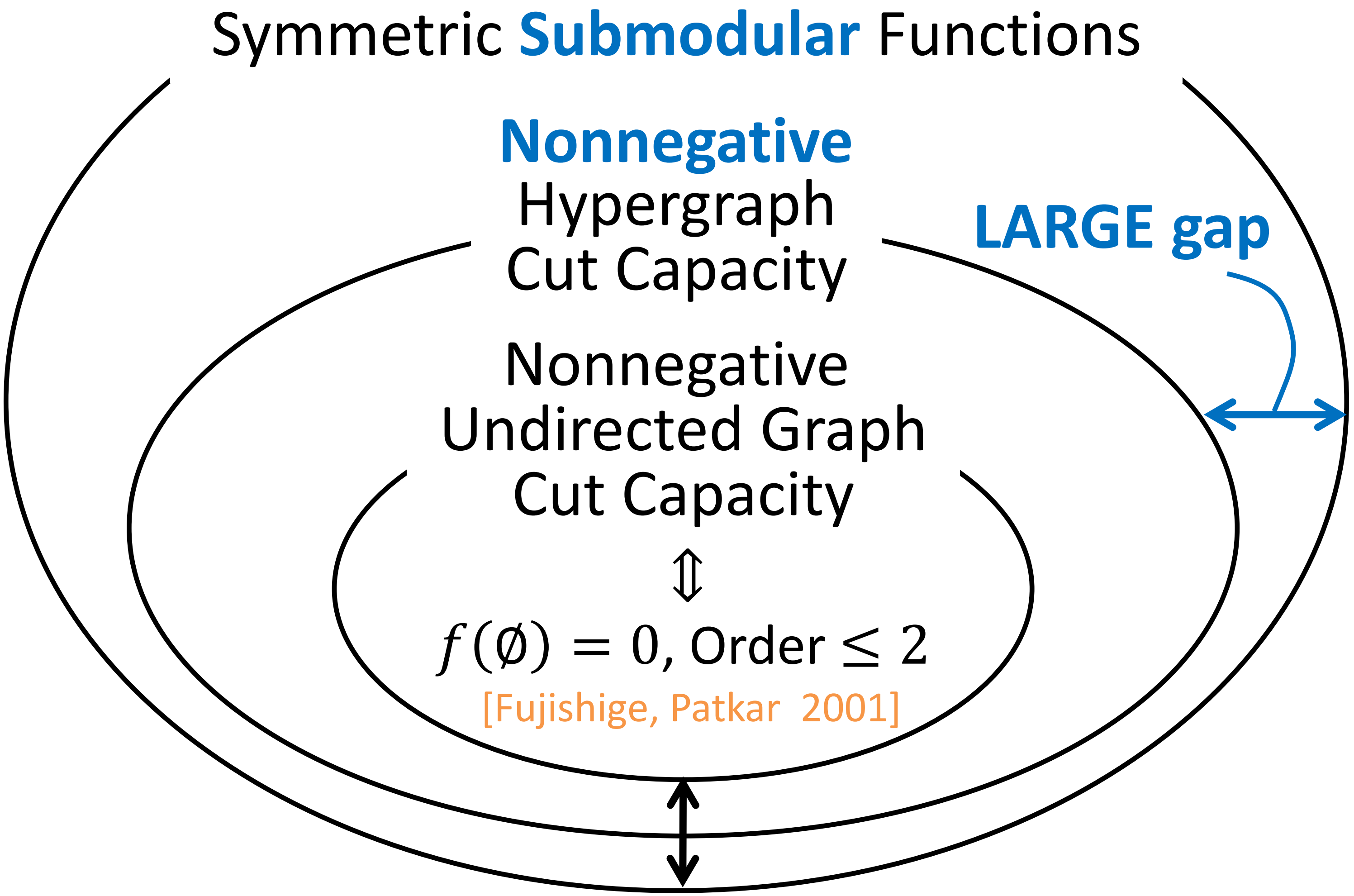

Symmetric **Submodular** Functions
Nonnegative
Hypergraph
Cut Capacity
LARGE gap
Nonnegative
Undirected Graph
Cut Capacity
⇕
$f(\emptyset) = 0$, Order ≤ 2
[Fujishige, Patkar 2001]

Realizability by Hypergraph

$f: 2^V \to \mathbf{R}$, symmetric, submodular

f is **realizable** as the cut capacity function of a **nonnegative hypernetwork**. $(\forall E \in \mathcal{E},\ c(E) \geq 0)$

$$\Downarrow \ \not\Uparrow$$

$$f(\emptyset) = 0.$$

$\exists$ Counterexample with $|V| = 4$ for $\Uparrow$

LARGE Gap!

Realizability by Hypergraph

$f: 2^V \to \mathbf{R}$, symmetric, submodular

f is **realizable** as the cut capacity function of a **nonnegative hypernetwork**. $(\forall E \in \mathcal{E},\ c(E) \geq 0)$

$$\Downarrow \ \not\Uparrow$$

$$f(\emptyset) = 0,$$

NEW! the **even-order** terms of f are **nonpositive**, the **odd-order** terms of f are **nonnegative**.

[Y. 2015]

$\exists$ Counterexample with $|V| = 5$ for $\Uparrow$

Still LARGE Gap!!

Put **nonnegativity** aside ...

Redundancy of Hypergraph Realization

$f: 2^V \to \mathbf{R}$, symmetric

f is **realizable** as the cut cap. func. of a **hypernetwork**.

$$\Updownarrow$$

$$f(\emptyset) = 0.$$

Corollary of [Grishuhin 1989] (Reminder)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \mid f(\emptyset) = 0, \;\; f(X) = f(V \setminus X) \;\; (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork $\left(c \in \mathbf{R}^{\mathcal{E}}\right)$

$$\downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1, \;\; \dim \mathbf{R}^{\mathcal{E}} = |\mathcal{E}| \leq 2^{|V|}$$

Non-Redundant Hypergraphs?

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \mid f(\emptyset) = 0, \quad f(X) = f(V \setminus X) \ \ (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$$\Downarrow$$

The linear mapping $c \mapsto \kappa_{\mathcal{N}}$ $\left(\mathbf{R}^{\mathcal{E}} \to \mathcal{F}\right)$ can be **bijective**.

The cut capacity function $\kappa_{\mathcal{N}}: 2^V \to \mathbf{R}$

$$\kappa_{\mathcal{N}}(X) := \sum_{E \in \mathcal{E}} \{ c(E) \mid E \cap X \neq \emptyset \neq E \setminus X \}$$

(Reminder)

Standard Form 1 (Rooted)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \mid f(\emptyset) = 0, \quad f(X) = f(V \setminus X) \ \ (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$r \in V,\ \mathcal{E} = \{ X \mid r \in X \subseteq V,\ |X| \geq 2 \}$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$\because$ $X \in \mathcal{E} \iff \emptyset \neq \exists Z \subseteq V - r$ s.t. $X = Z + r$

#(choices of Z) $= 2^{|V-r|} - 1$

Standard Form 1 (Rooted)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \;\middle|\; f(\emptyset) = 0, \quad f(X) = f(V \setminus X) \ \ (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$$r \in V,\ \mathcal{E} = \{\, X \mid r \in X \subseteq V,\ |X| \geq 2 \,\}$$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$$r \in V,\ \mathcal{E} = \{\, X \mid r \in X \subseteq V,\ |X| \geq 2 \,\}$$

$$\Downarrow$$

The linear mapping $c \mapsto \kappa_{\mathcal{N}}$ $\left(\mathbf{R}^{\mathcal{E}} \to \mathcal{F}\right)$ is **bijective**.

[Y. 2015]

Standard Form 2 (Even-size)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \mid f(\emptyset) = 0,\ \ f(X) = f(V \setminus X)\ \ (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$V \neq \emptyset,\ \mathcal{E} = \{ X \mid \emptyset \neq X \subseteq V,\ |X|\text{: even} \}$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$$\because\ 2^{|V|} = (1+1)^{|V|} + (1-1)^{|V|} = \sum_{X \subseteq V} \left(1 + (-1)^{|X|}\right)$$

$$= 2|\{ X \subseteq V \mid |X|\text{: even} \}| = 2(|\mathcal{E}| + 1)$$

Standard Form 2 (Even-size)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \;\middle|\; f(\emptyset) = 0, \quad f(X) = f(V \setminus X) \;\; (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$V \neq \emptyset,\ \mathcal{E} = \{\, X \mid \emptyset \neq X \subseteq V,\ |X|\text{: even} \,\}$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$V \neq \emptyset,\ \mathcal{E} = \{\, X \mid \emptyset \neq X \subseteq V,\ |X|\text{: even} \,\}$

$$\Downarrow$$

The linear mapping $c \mapsto \kappa_{\mathcal{N}}$ $\left(\mathbf{R}^{\mathcal{E}} \to \mathcal{F}\right)$ is **bijective**.

[Grishuhin 1989] Alternative Proof by [Y. 2015]

Standard Form 3 (Majority)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \;\middle|\; f(\emptyset) = 0, \;\; f(X) = f(V \setminus X) \;\; (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E} \setminus \{E\}), c)$: hypernetwork

$|V|$: odd, $\mathcal{E} = \left\{ X \subseteq V \mid \left\lceil \frac{|V|}{2} \right\rceil \leq |X| \leq |V| \right\}$, $E \in \mathcal{E}$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| - 1 = \dim \mathbf{R}^{\mathcal{E} \setminus \{E\}}$$

$$\because \; X \subseteq V, \qquad X \in \mathcal{E} \;\Leftrightarrow\; V \setminus X \notin \mathcal{E}$$

Standard Form 3 (Majority)

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$$\Downarrow$$

The linear mapping $c \mapsto \kappa_{\mathcal{N}} \;\left(\mathbf{R}^{\mathcal{E} \setminus \{E\}} \to \mathcal{F}\right)$ is **bijective**.

[Y. 2015]

How to see the correctness?

Rooted Standard Forms (Reminder)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \mid f(\emptyset) = 0,\ \ f(X) = f(V \setminus X)\ \ (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$$r \in V,\ \mathcal{E} = \{ X \mid r \in X \subseteq V,\ |X| \geq 2 \}$$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

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The linear mapping $c \mapsto \kappa_{\mathcal{N}}$ $\left(\mathbf{R}^{\mathcal{E}} \to \mathcal{F}\right)$ is **bijective**.

[Y. 2015]

Correctness of Rooted Standard Forms

$$\begin{bmatrix} \kappa_{\mathcal{N}}(\{r\}) \\ \kappa_{\mathcal{N}}(\{r, v_1\}) \\ \kappa_{\mathcal{N}}(\{r, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(\{r, v_1, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(V - v_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & & 1 & & 1 \\ 1 & 0 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & 1 & 0 & & 1 & & 1 \\ & \vdots & & \ddots & & & \vdots \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 1 \\ & \vdots & & & & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} c(V) \\ c(\{r, v_1\}) \\ c(\{r, v_2\}) \\ \vdots \\ c(\{r, v_1, v_2\}) \\ \vdots \\ c(V - v_1) \end{bmatrix}$$

$$r \in V,\ \mathcal{E} = \{\, X \mid r \in X \subseteq V,\ |X| \geq 2 \,\}$$

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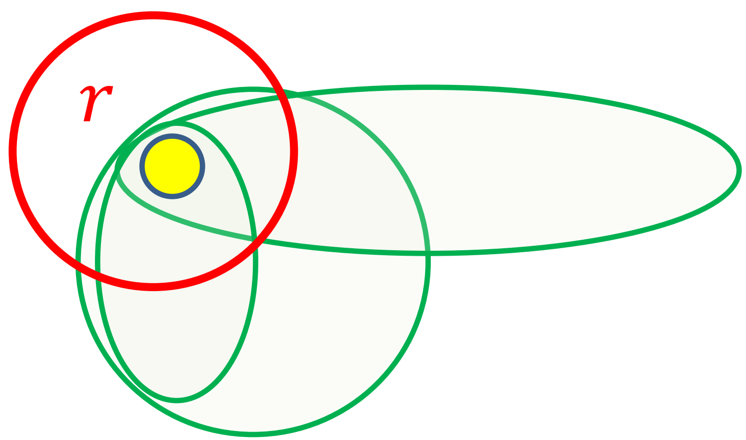

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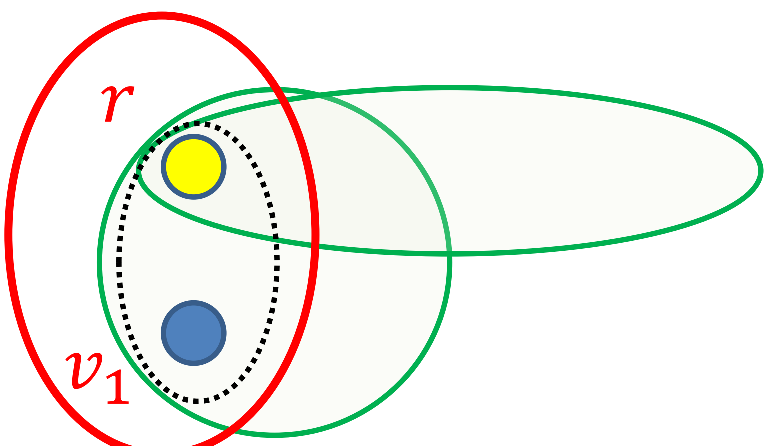
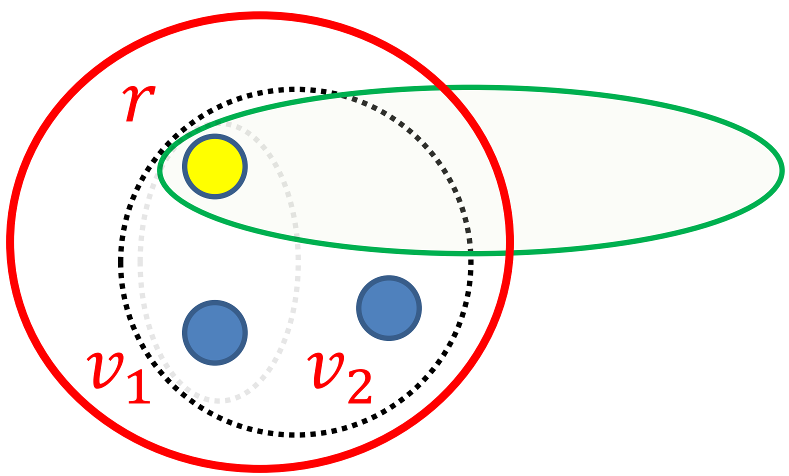


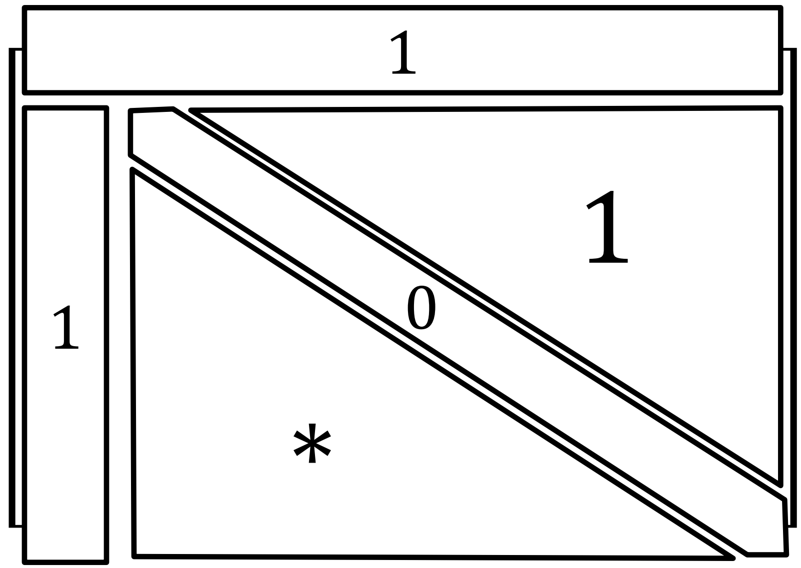
Correctness of Rooted Standard Forms

$$
\begin{bmatrix} \kappa_{\mathcal{N}}(\{r\}) \\ \kappa_{\mathcal{N}}(\{r, v_1\}) \\ \kappa_{\mathcal{N}}(\{r, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(\{r, v_1, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(V - v_1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & & 1 & & 1 \\
1 & 0 & 1 & \cdots & 1 & \cdots & 1 \\
1 & 1 & 0 & & 1 & & 1 \\
 & \vdots & & \ddots & & & \vdots \\
1 & 0 & 0 & \cdots & 0 & \cdots & 1 \\
 & \vdots & & & & \ddots & \vdots \\
1 & 1 & 0 & \cdots & 1 & \cdots & 0
\end{bmatrix}
\begin{bmatrix} c(V) \\ c(\{r, v_1\}) \\ c(\{r, v_2\}) \\ \vdots \\ c(\{r, v_1, v_2\}) \\ \vdots \\ c(V - v_1) \end{bmatrix}
$$



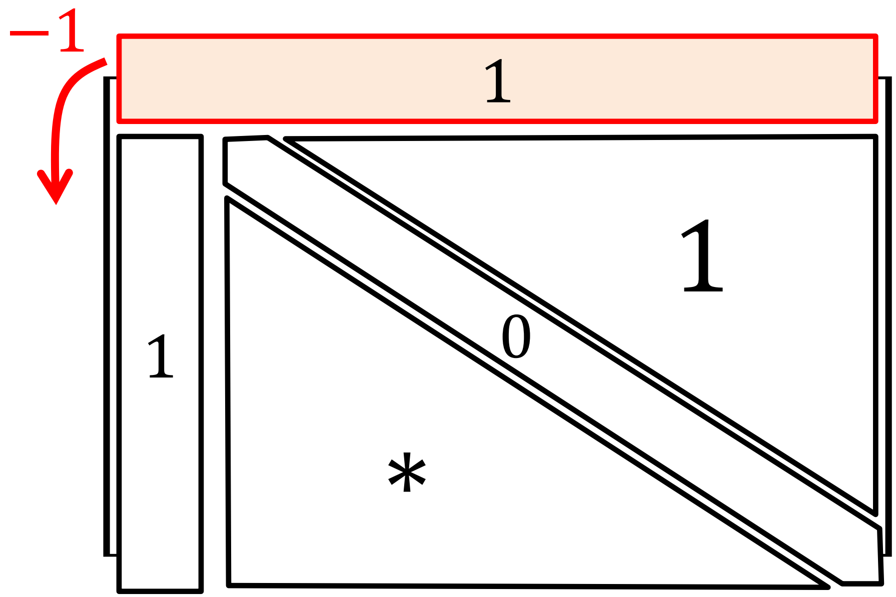
Correctness of Rooted Standard Forms

$$\begin{bmatrix} \kappa_{\mathcal{N}}(\{r\}) \\ \kappa_{\mathcal{N}}(\{r, v_1\}) \\ \kappa_{\mathcal{N}}(\{r, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(\{r, v_1, v_2\}) \\ \vdots \\ \kappa_{\mathcal{N}}(V - v_1) \end{bmatrix} =$$

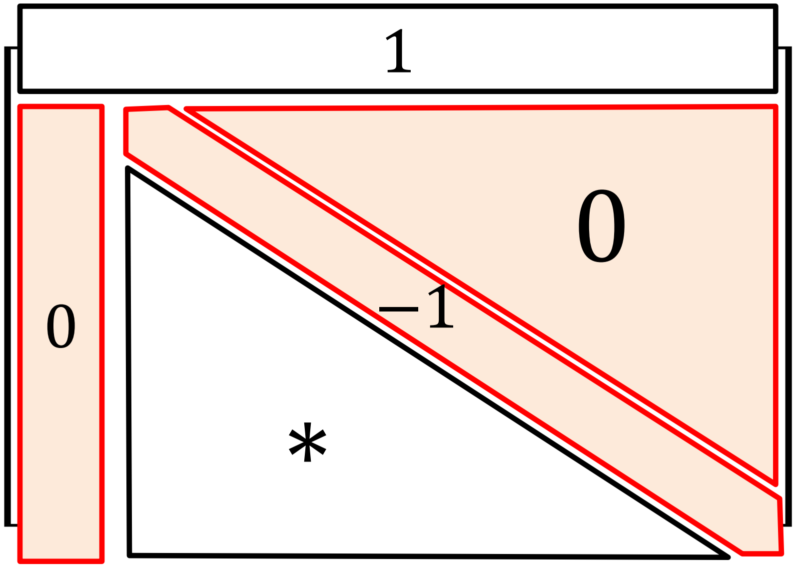


$$\begin{bmatrix} c(V) \\ c(\{r, v_1\}) \\ c(\{r, v_2\}) \\ \vdots \\ c(\{r, v_1, v_2\}) \\ \vdots \\ c(V - v_1) \end{bmatrix}$$

Correctness of Rooted Standard Forms



Correctness of Rooted Standard Forms



Nonsingular

Even Standard Forms (Reminder)

$$\mathcal{F} := \left\{ f \in \mathbf{R}^{2^V} \;\middle|\; f(\emptyset) = 0, \;\; f(X) = f(V \setminus X) \;\; (\forall X \subseteq V) \right\}$$

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$V \neq \emptyset, \; \mathcal{E} = \{ X \mid \emptyset \neq X \subseteq V, \; |X|\text{: even} \}$

$$\Downarrow$$

$$\dim \mathcal{F} = 2^{|V|-1} - 1 = |\mathcal{E}| = \dim \mathbf{R}^{\mathcal{E}}$$

$V \neq \emptyset, \; \mathcal{E} = \{ X \mid \emptyset \neq X \subseteq V, \; |X|\text{: even} \}$

$$\Downarrow$$

The linear mapping $c \mapsto \kappa_{\mathcal{N}} \; \left(\mathbf{R}^{\mathcal{E}} \to \mathcal{F} \right)$ is **bijective**.

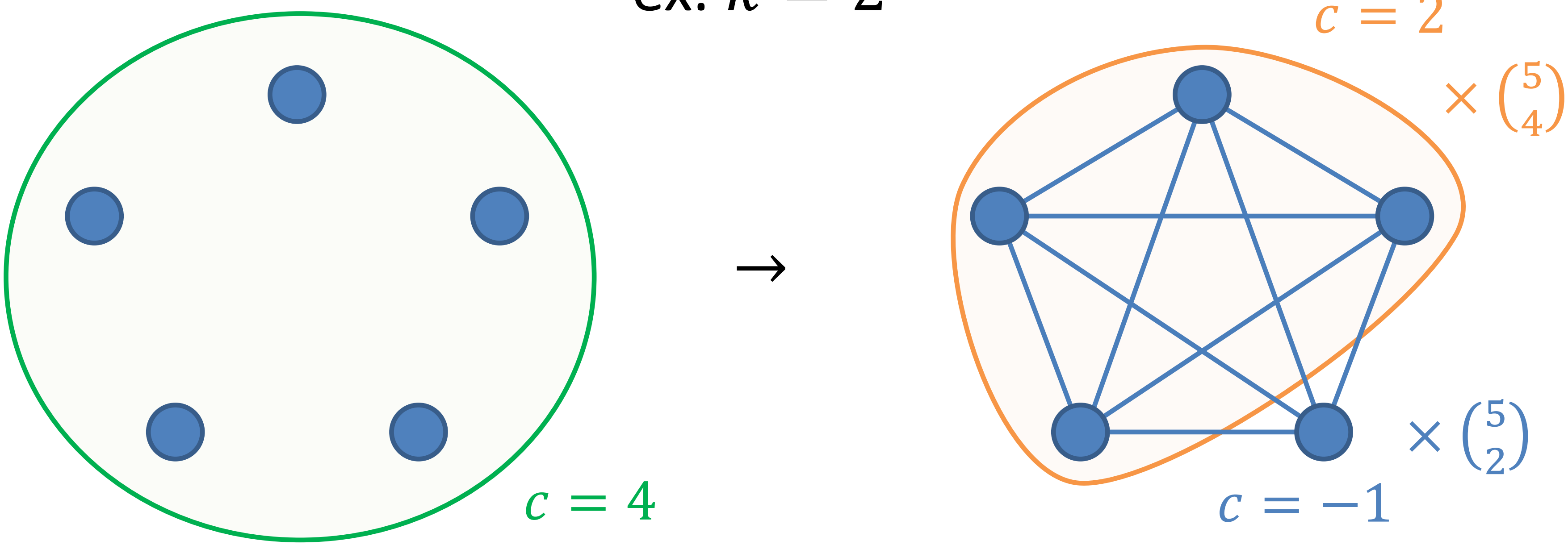
[Grishuhin 1989] Alternative Proof by [Y. 2015]

Odd-size Hyperedges

For $k \in \mathbf{Z}_{>0}$, any hyperedge of size $2k + 1$ **can be replaced** by ones of size $2, 4, \ldots, 2k$.

[Y. 2015]

ex. $k = 2$



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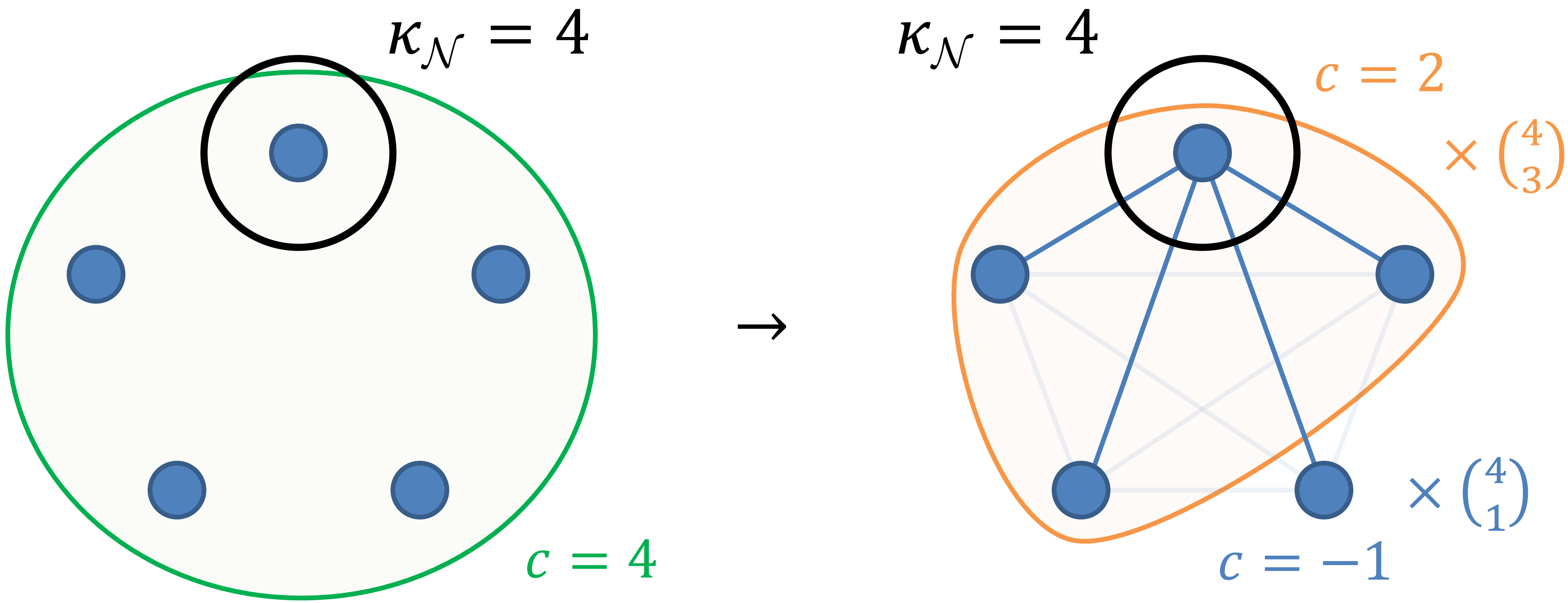

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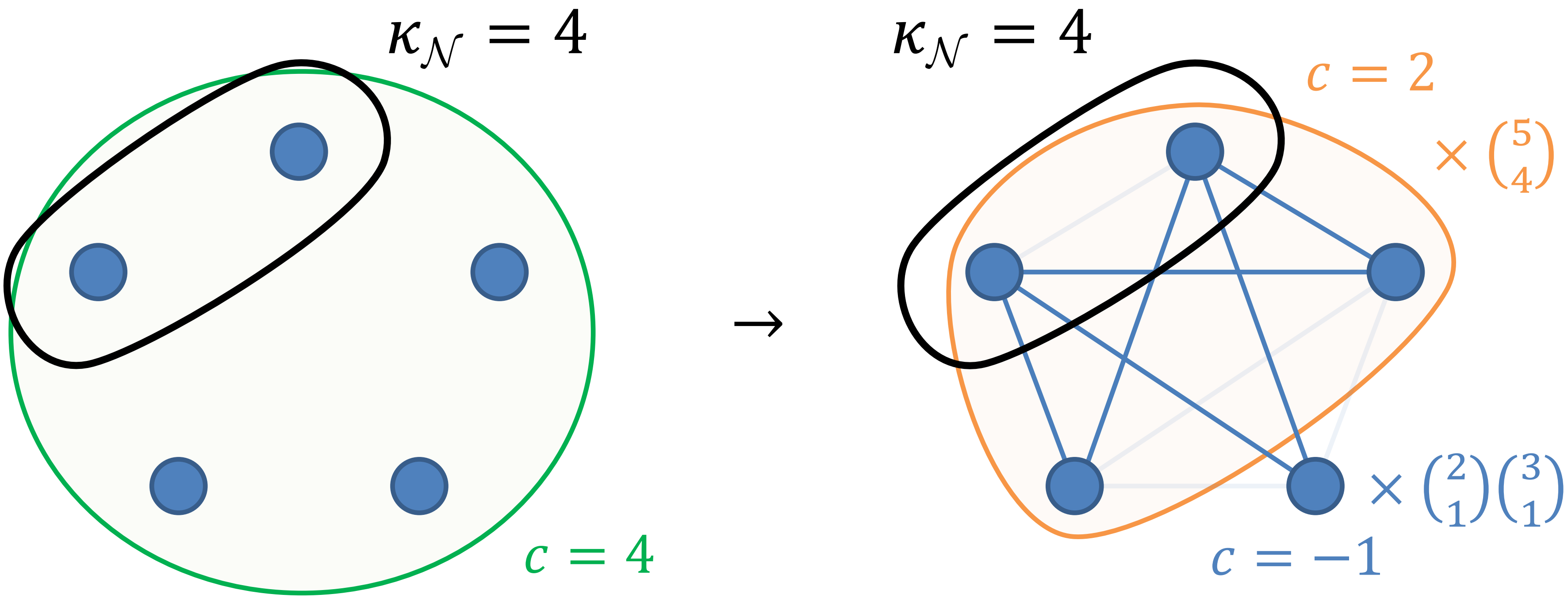

Conclusion

- Any symmetric real-valued set function f with $f(\emptyset) = 0$ can be realized as **cut capacity of a hypergraph**.

(Extends the case of undirected graphs [Fujishige, Patkar 2001])

- We give three types of hyperedge sets consisting **bases for cut realization** as **standard forms**.

(with a fixed root, even-size, majorities without any one)
[Grishuhin 1989]

- For the case when the capacity function is **nonnegative**, sufficient conditions are still **OPEN** ...

Properties of Cut Capacity Functions

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

- $\kappa_{\mathcal{N}}$ is **symmetric**, *i.e.*,

$$\kappa_{\mathcal{N}}(X) = \kappa_{\mathcal{N}}(V \setminus X) \quad (X \subseteq V)$$

- $c: \mathcal{E} \to \mathbf{R}$ is **nonnegative** $(\forall E \in \mathcal{E},\ c(E) \geq 0)$

  $\Rightarrow$ $\kappa_{\mathcal{N}}$ is **submodular**, *i.e.*,

$$\kappa_{\mathcal{N}}(X) + \kappa_{\mathcal{N}}(Y) \geq \kappa_{\mathcal{N}}(X \cup Y) + \kappa_{\mathcal{N}}(X \cap Y) \quad (X, Y \subseteq V)$$

From the Viewpoint of Minimization

$f : 2^V \to \mathbf{R}$, **symmetric** and **submodular**

$$X^* \in \operatorname*{argmin}_{X : \emptyset \neq X \subset V} f(X)$$

can be found in $\mathrm{O}(|V|^3 \mathrm{EO})$ time (EO: eval. cost of f)

[Queyranne 1998]

- Generalizes minimum-cut algorithms for
  - undirected graphs [Nagamochi, Ibaraki 1992] and
  - hypergraphs [Klimmek, Wagner 1996].
- Solved by repeated general submodular minimizations, requiring $\mathrm{O}(|V|^5 \mathrm{EO} + |V|^6)$ time per once [Orlin 2009].

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→ **How close** the set of **cut capacity functions** is to the set of **symmetric submodular functions**?

Order of Set Function

$\forall f: 2^V \to \mathbf{R}$, $\exists! F \in \mathbf{R}[x_v \mid v \in V]$: polynomial s.t.

$F(\mathbf{1}_X) = f(X)$ $(X \subseteq V)$, and

$$F(x) = \sum_{X \subseteq V} a_X \prod_{v \in X} x_v \quad \big(x = (x_v \mid v \in V)\big)$$

(cf. Möbius Inversion Formula)

- (The order of f) $:= \deg F = \max_{X \subseteq V}\{|X| \mid a_X \neq 0\}$.
- a_X: $|X|$-th order term

Odd(Even)-Order Terms of Cut Cap.

$\mathcal{N} = (\mathcal{H} = (V, \mathcal{E}), c)$: hypernetwork

$$a_X = \begin{cases} \sum_{E \in \mathcal{E}} \{ c(E) \mid X \subset E \} & (|X|\text{: odd}) \\ -\left(2c(X) + \sum_{E \in \mathcal{E}} \{ c(E) \mid X \subset E \} \right) & (|X|\text{: even}) \end{cases}$$

[Y. 2015]

$F(\mathbf{1}_X) = \kappa_{\mathcal{N}}(X) \qquad (X \subseteq V)$, and

$$F(x) = \sum_{X \subseteq V} a_X \prod_{v \in X} x_v \quad \left(x = (x_v \mid v \in V) \right)$$

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