Exact Matching in Matrix Multiplication Time

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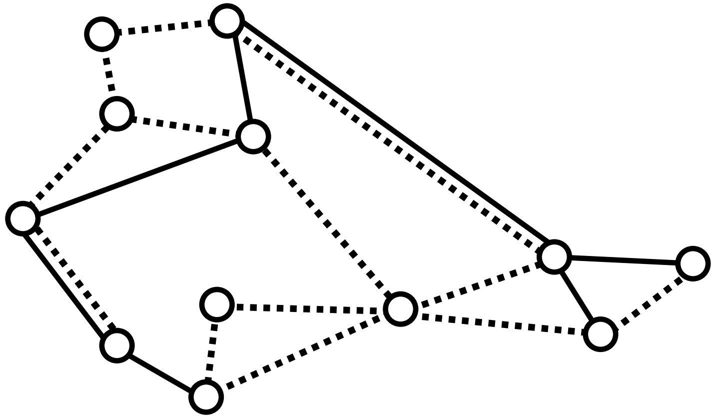
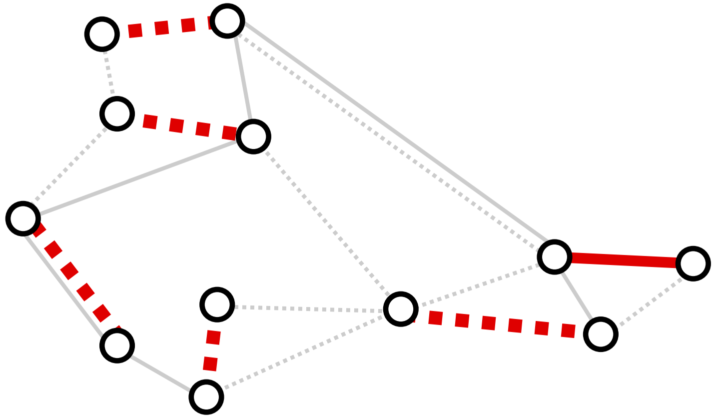
Exact Matching Problem (EM) [Papadimitriou–Yannakakis 1982]

Input: $G = (V, E)$: Undirected Graph, $w: E \to \{0, 1\}$, $k \in \mathbf{Z}$

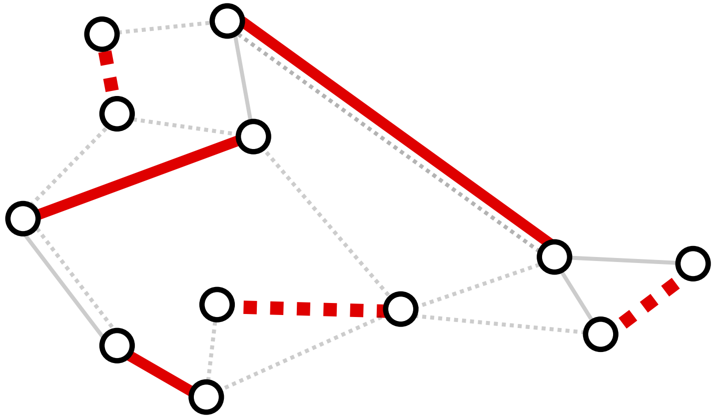
Question: Does a **Perfect Matching** $M \subseteq E$ with $w(M) = k$ exist?



$$w(e) = \begin{cases} 1 \\ 0 \end{cases}$$



$w(M) = 1$



$w(M) = 3$

Exact Matching Problem (EM) [Papadimitriou–Yannakakis 1982]

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Question: Does a **Perfect Matching** $M \subseteq E$ with $w(M) = k$ exist?

- **Randomized** Polytime Algorithm **in general** [Mulmuley–Vazirani–Vazirani 1987]

vs.

- **Deterministic** Polytime Algorithm **for very limited cases**
  [Karzanov 1987; Vazirani 1989, Yuster 2012; Galluccio–Loebl 1999; ...]

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Thm. One can test, for every k at once, whether an EM exists or not in $\mathrm{O}\left(n^\omega \mathrm{poly}(\log n)\right)$ time (field operations) in total. $(\omega < 2.37134)$

e.g., [Camerini–Galbiati–Maffioli 1992] + [Storjohann 2003]

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[This work]

e.g., [Camerini–Galbiati–Maffioli 1992] + [Storjohann 2003]

Idea: Reduce to computing the Characteristic Polynomial $\det(tI - A)$

Outline

- Basics: Matching and Tutte Matrix
- An $O(n^\omega)$-time Randomized Algorithm for Perfect Matching (Existence)
- An $O(n^\omega)$-time Randomized Algorithm for Exact Matching (Existence)
- Remarks and Open Questions

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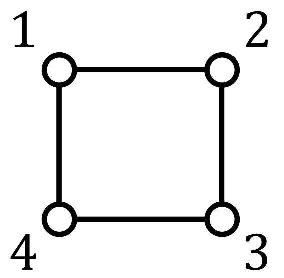
Tutte Matrix

$\mathbf{F}$: field (e.g., $\mathrm{GF}(p)$ for some prime p)

The Tutte matrix $T(G)$ of $G = (V, E)$ is a $V \times V$ matrix defined as follows:

- Fix a total order on V
- $X_E \coloneqq \{\, x_e \mid e \in E \,\}$: indeterminates
- $T(G)_{u,v} \coloneqq \begin{cases} x_e & e = \{u, v\} \in E,\ u < v \\ -x_e & e = \{u, v\} \in E,\ u > v \\ 0 & \{u, v\} \notin E \end{cases}$



$$\begin{bmatrix} 0 & x_{12} & 0 & x_{14} \\ -x_{12} & 0 & x_{23} & 0 \\ 0 & -x_{23} & 0 & x_{34} \\ -x_{14} & 0 & -x_{34} & 0 \end{bmatrix}$$

Tutte Matrix

$(\mathrm{pf}\,A)^2 \equiv \det A$

Thm. G has a Perfect Matching $\Leftrightarrow$ $\mathrm{pf}\,T(G) \not\equiv 0$ $\Leftrightarrow$ $\det T(G) \not\equiv 0$

$$\sum_{M:\,\text{perfect matching}} \mathrm{sgn}(M) \prod_{e \in M} x_e$$

[Tutte 1947]

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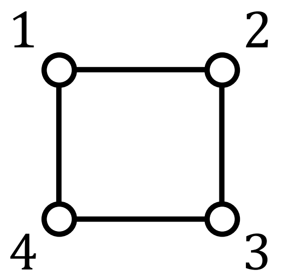

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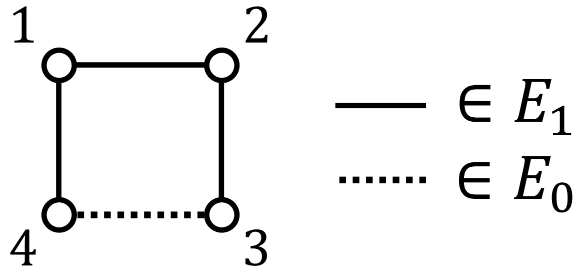
Tutte Matrix (Weighted)

(G_0, G_1): EM instance (0/1-edge-weighted graph), where
$G_i = (V, E_i)$ is the subgraph formed by edges of weight i

The Tutte matrix of (G_0, G_1) is defined as follows:

- y: extra indeterminate
- $T(G_0, G_1) \coloneqq T(G_0) + yT(G_1)$



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Tutte Matrix (Weighted)

Thm. (G_0, G_1) has a Perfect Matching of weight exactly k

$\Leftrightarrow$ $[y^k]\,\mathrm{pf}\,T(G_0, G_1) \not\equiv 0$ (coeff. of y^k as a polynomial of x_e-s)

$$\sum_{M\text{: perfect matching}} \mathrm{sgn}(M) \prod_{e \in M} y^{w(e)} x_e$$

e.g., [MVV1987, CGM1992]

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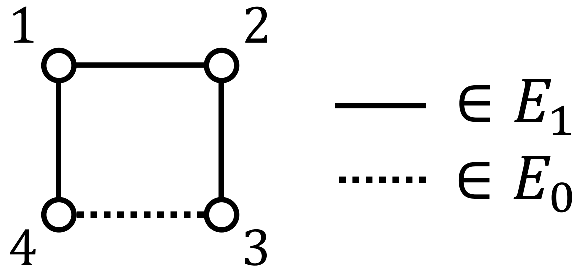

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Perfect/Exact Matching via Tutte Matrix

Thm.

- G has a Perfect Matching $\Leftrightarrow$ $\mathrm{pf}\, T(G) \not\equiv 0$ $\Leftrightarrow$ $\det T(G) \not\equiv 0$
- (G_0, G_1) has a Perfect Matching of weight exactly k
  $\Leftrightarrow$ $[y^k]\, \mathrm{pf}\, T(G_0, G_1) \not\equiv 0$ (coeff. of y^k as a polynomial of x_e-s)

Generally, the problems reduce to **PIT (Polynomial Identity Testing)**

- **Deterministic** computation is difficult (at least unknown)
- **Randomized** computation is easy when the field $\mathbf{F}$ is sufficiently large

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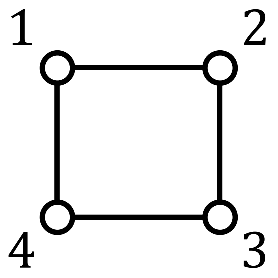
Perfect Matching via Tutte Matrix

Input: $G = (V, E)$: Undirected Graph

Question: Does a Perfect Matching $M \subseteq E$ exist?



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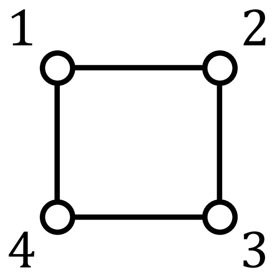
Thm. G has a Perfect Matching $\iff \det T(G) \not\equiv 0$

Difficult to compute $\det T(G) \in \mathbf{F}[X_E]$ (as a polynomial of x_e-s)

- After substituting any specific value $\tilde{x}_e \in \mathbf{F}$ to each x_e, one can compute $\det \tilde{T}(G) \in \mathbf{F}$ in $O(n^\omega)$ time **(deterministically)**
- When $|\mathbf{F}|$ is large, $\det T(G) \not\equiv 0 \iff \det \tilde{T}(G) \neq 0$ **with high prob.**

Perfect Matching via Tutte Matrix



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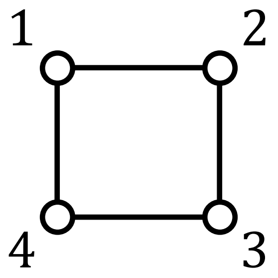

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Thm. G has a Perfect Matching $\Leftrightarrow$ $\det T(G) \not\equiv 0$

Thm. f is a nonzero polynomial of x_i $(i \in [m])$ of total degree d, and r_i $(i \in [m])$ is chosen uniformly at random from $S \subseteq \mathbf{F}$

$$\Longrightarrow \quad \Pr[f(r_1, \dots, r_m) = 0] \leq \frac{d}{|S|}$$

(Schwartz–Zippel Lemma)

$\mathbf{F} = \mathrm{GF}(p)$ $(p \gg n^2)$ is enough to test with prob. $1 - n^{-1}$ in $O(n^\omega)$ time

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Exact Matching via Tutte Matrix

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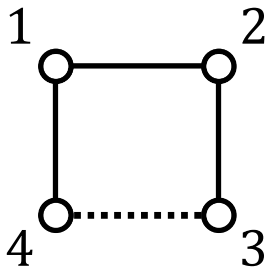
Question: Does a Perfect Matching $M \subseteq E$ with $w(M) = k$ exist?

__**Thm.**__ (G_0, G_1) has a Perfect Matching of weight exactly k
$\iff$ $[y^k]\,\mathrm{pf}\,T(G_0, G_1) \not\equiv 0$ (coeff. of y^k as a polynomial of x_e-s)



——— $\in E_1 = \{ e \in E \mid w(e) = 1 \}$
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Almost the same approach works in $\mathrm{O}(n^{\omega+1})$ time

- After random substitution to x_e-s, compute $\det \tilde{T}(G_0, G_1) \in \mathbf{F}[y]$ by polynomial interpolation with evaluation at $y = 0, 1, \dots, n$
- Reconstruct $\mathrm{pf}\,\tilde{T}(G_0, G_1) \in \mathbf{F}[y]$ (up to sign) using $(\mathrm{pf}\,A)^2 \equiv \det A$

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Only bottleneck

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Compute $\det \tilde{T}(G_0, G_1) \in \mathbf{F}[y]$ after random substitution to x_e-s

Thm. For $A \in \mathbf{F}[y]^{n\times n}$ s.t. $\deg a_{ij} \le d \ (\forall i,j)$,
$\det A \in \mathbf{F}[y]$ is computed in $\mathrm{O}\big(n^\omega d \operatorname{poly}(\log n + \log d)\big)$ time w.h.p.

[Storjohann 2003]

- Direct application of this $\rightarrow$ $\mathrm{O}\big(n^\omega \operatorname{poly}(\log n)\big)$ time w.h.p. (Las Vegas)
- We reduce the task to computing the Characteristic Polynomial $\det(tI - A)$

Thm. For $A \in \mathbf{F}^{n\times n}$,
$\det(tI - A) \in \mathbf{F}[t]$ is computed in $\mathrm{O}(n^\omega)$ time deterministically

[Neiger–Pernet 2021]

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$\boxed{\text{Compute } \det \tilde{T}(G_0, G_1) \in \mathbf{F}[y] \text{ after random substitution to } x_e\text{-s}}$

We reduce it to computing the Characteristic Polynomial $\det(tI - A)$

$$\tilde{T}(G_0, G_1) \equiv \tilde{T}(G_0) + y\tilde{T}(G_1) \qquad \text{(Definition)}$$

$$\equiv \tilde{T}(G) + (y-1)\tilde{T}(G_1) \qquad (G = G_0 + G_1)$$

$$\equiv \tilde{T}(G)\left(I + (y-1)\tilde{T}(G)^{-1}\tilde{T}(G_1)\right) \qquad (G \text{ should have PM})$$

$$\equiv (y-1)\tilde{T}(G)\left(tI - \left(-\tilde{T}(G)^{-1}\tilde{T}(G_1)\right)\right) \qquad \left(t := \frac{1}{y-1}\right)$$

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$$\equiv \tilde{T}(G)\left(I + (y-1)\tilde{T}(G)^{-1}\tilde{T}(G_1)\right) \qquad (G \text{ should have PM})$$

$$\equiv (y-1)\tilde{T}(G)\left(tI - \left(-\tilde{T}(G)^{-1}\tilde{T}(G_1)\right)\right) \qquad \left(t \coloneqq \frac{1}{y-1}\right)$$

Bottleneck of EM via Tutte Matrix

$$\begin{bmatrix} 0 & y\tilde{x}_{12} & 0 & y\tilde{x}_{14} \\ -y\tilde{x}_{12} & 0 & y\tilde{x}_{23} & 0 \\ 0 & -y\tilde{x}_{23} & 0 & \tilde{x}_{34} \\ -y\tilde{x}_{14} & 0 & -\tilde{x}_{34} & 0 \end{bmatrix}$$

Compute $\det \tilde{T}(G_0, G_1) \in \mathbf{F}[y]$ after random substitution to x_e-s

We reduce it to computing the Characteristic Polynomial $\det(tI - A)$

$$\begin{aligned} \tilde{T}(G_0, G_1) &\equiv \tilde{T}(G_0) + y\tilde{T}(G_1) && \text{(Definition)} \\ &\equiv \tilde{T}(G) + (y-1)\tilde{T}(G_1) && (G = G_0 + G_1) \\ &\equiv \tilde{T}(G)\left(I + (y-1)\tilde{T}(G)^{-1}\tilde{T}(G_1)\right) && (G \text{ should have PM}) \\ &\equiv (y-1)\tilde{T}(G)\left(tI - \underbrace{\boxed{\left(-\tilde{T}(G)^{-1}\tilde{T}(G_1)\right)}}_{=:\,A}\right) && \left(t := \frac{1}{y-1}\right) \end{aligned}$$

$\rightarrow \quad \det \tilde{T}(G_0, G_1) \equiv (y-1)^n \det \tilde{T}(G) \det(tI - A)$

Outline

- Basics: Matching and Tutte Matrix
- An $O(n^{\omega})$-time Randomized Algorithm for Perfect Matching (Existence)
- An $O(n^{\omega})$-time Randomized Algorithm for Exact Matching (Existence)
- Remarks and Open Questions

Remarks and Open Questions

Exact Matching:
Perfect Matching of weight exactly k

Thm. One can test w.h.p., for every k at once, whether an **EM** exists or not in $O(n^{\omega})$ time (field operations) in total. $(\omega < 2.37134)$

Idea: Reduce to computing the Characteristic Polynomial $\det(tI - A)$

- For each possible k, an EM itself can be found in $O(n^{\omega+1})$ time by sequentially fixing $i_v \in \{0, 1\}$ $(v \in V)$ (which weight should be used)

  Q. Speeding-up? E.g., at once in $O(n^{\omega+1})$ time, or each in $O(n^{\omega})$ time

- A similar argument is applicable to Weighted Linear Matroid Parity, e.g., the min-length of a cycle through 3 specified vertices in $O(n^{\omega})$ time

  Q. Another application of this method?

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