Finding a Path in Group-Labeled Graphs with Two Labels Forbidden

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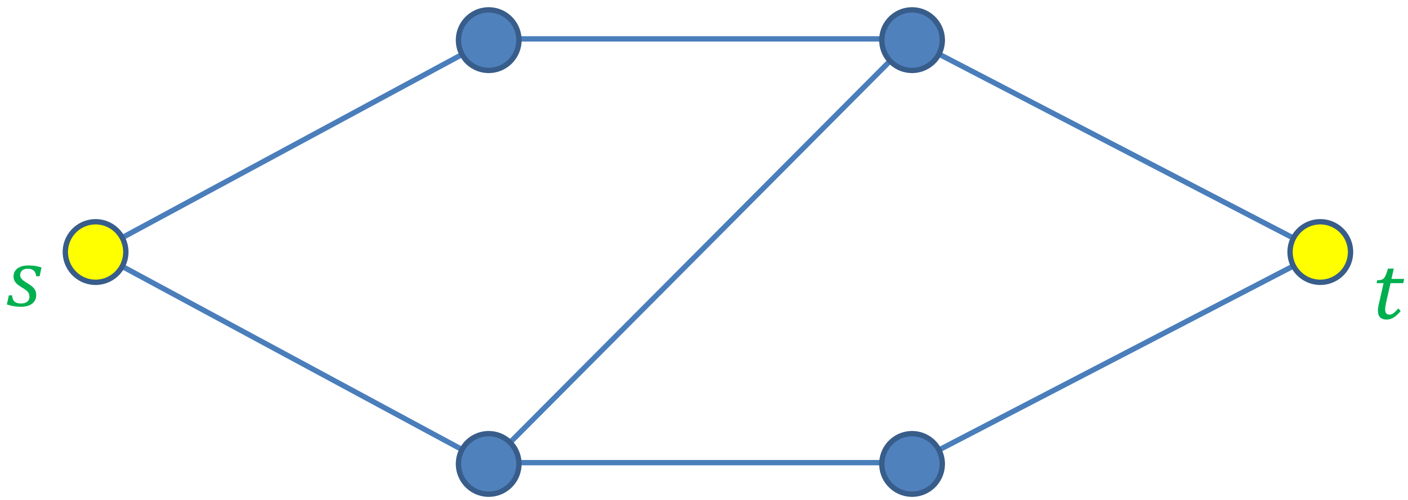
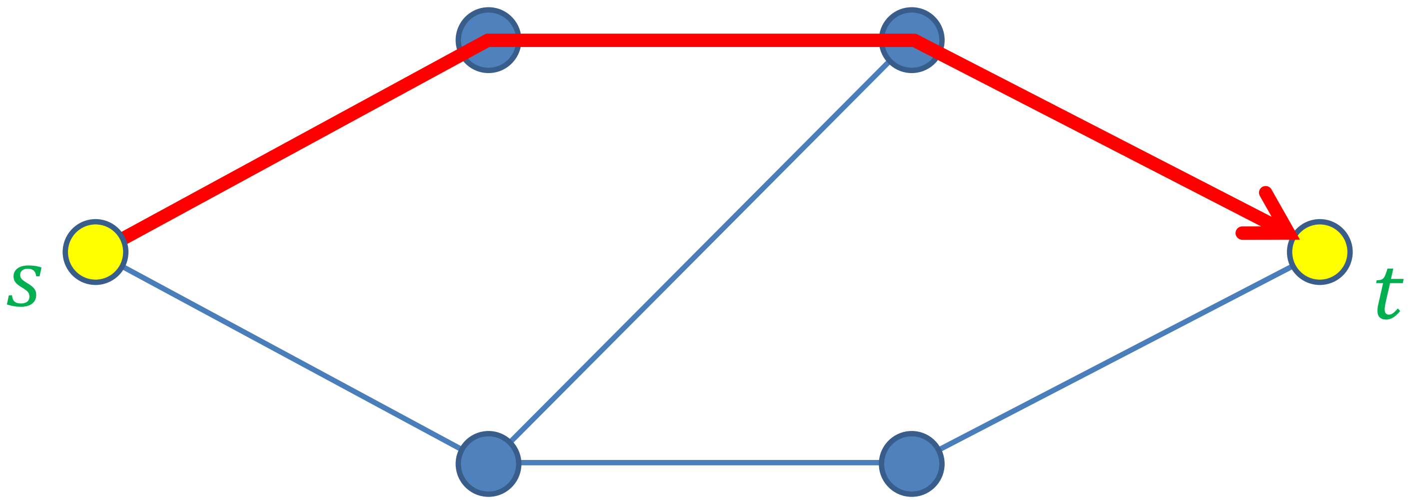
Parity of Length



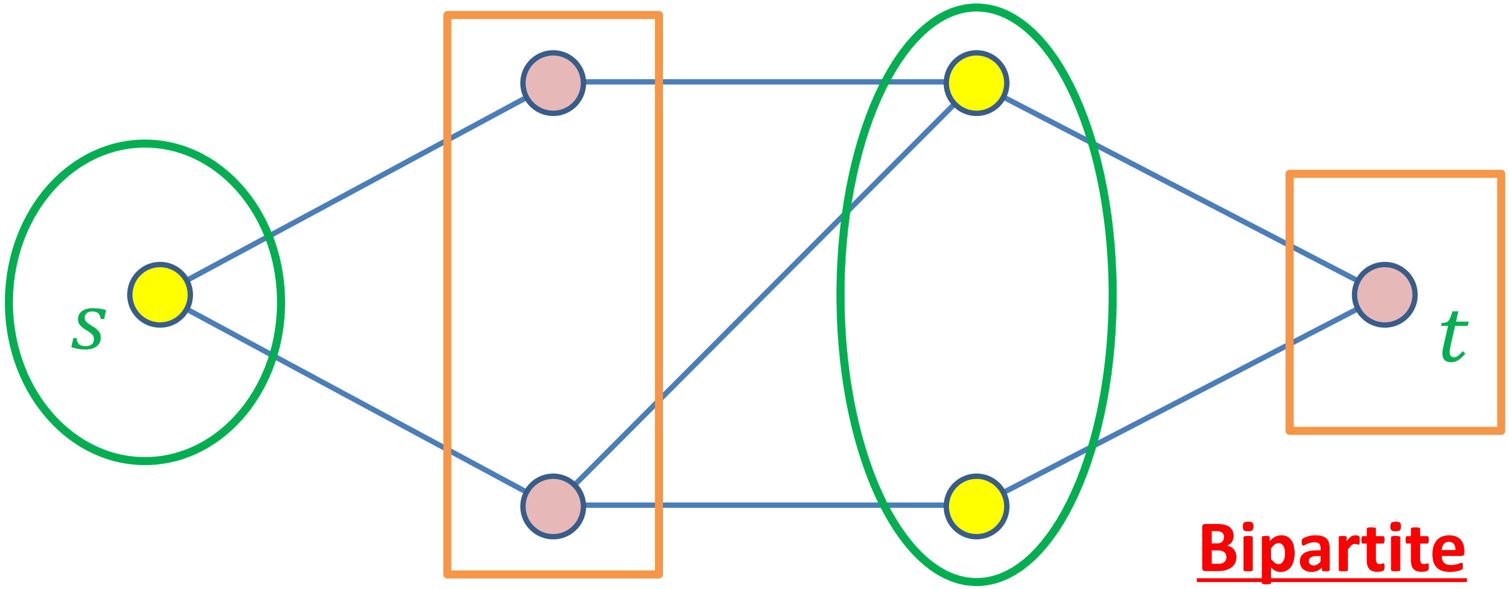
- Odd $\rightarrow$???
- Even $\rightarrow$???

Parity of Length



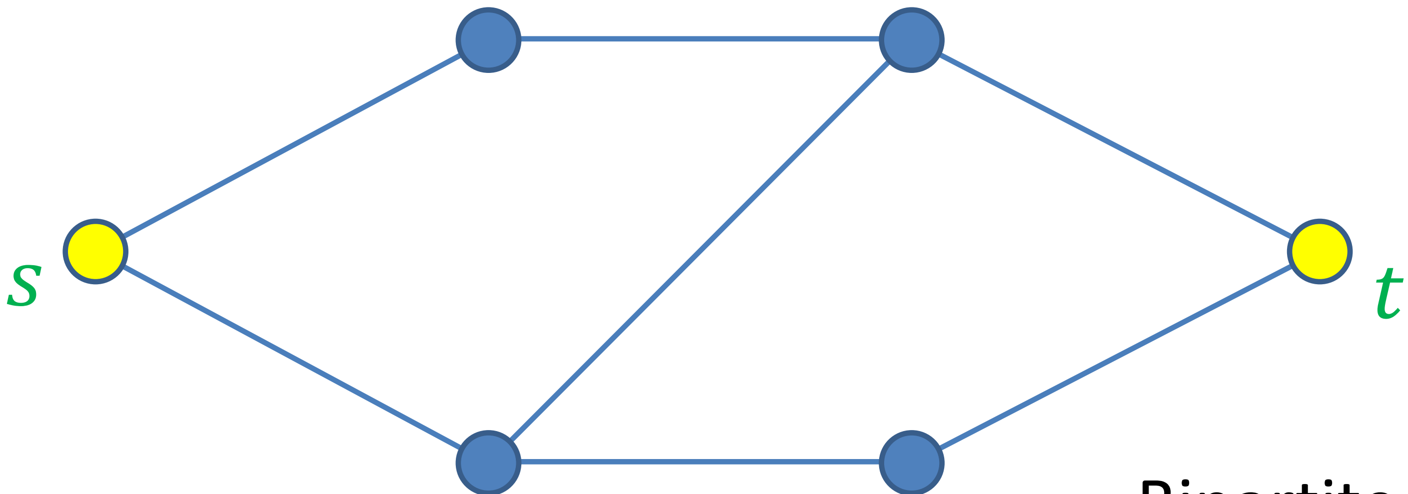
- **<u>Odd $\rightarrow$ YES</u>**
- Even $\rightarrow$???

Parity of Length



- Odd → YES
- **Even → NO**

Parity of Length



Bipartite

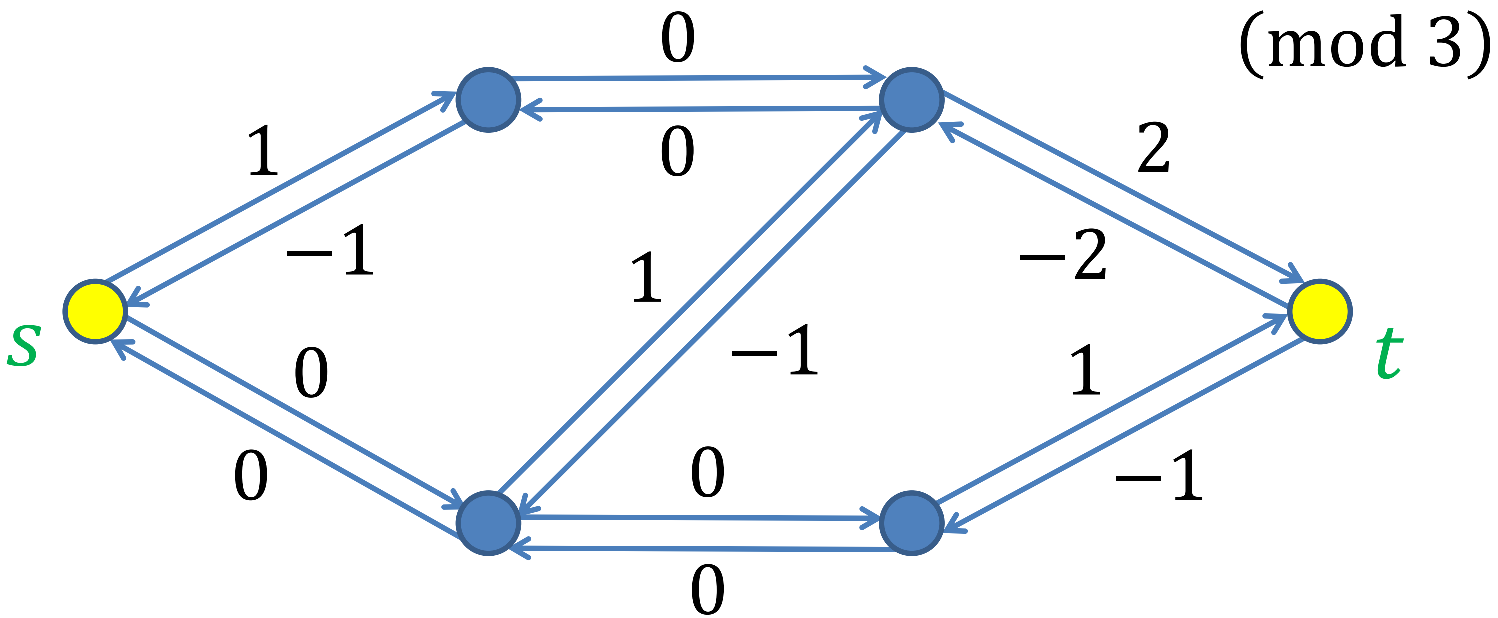
$\downarrow$

Easy to test!

- Odd $\rightarrow$ YES
- Even $\rightarrow$ NO

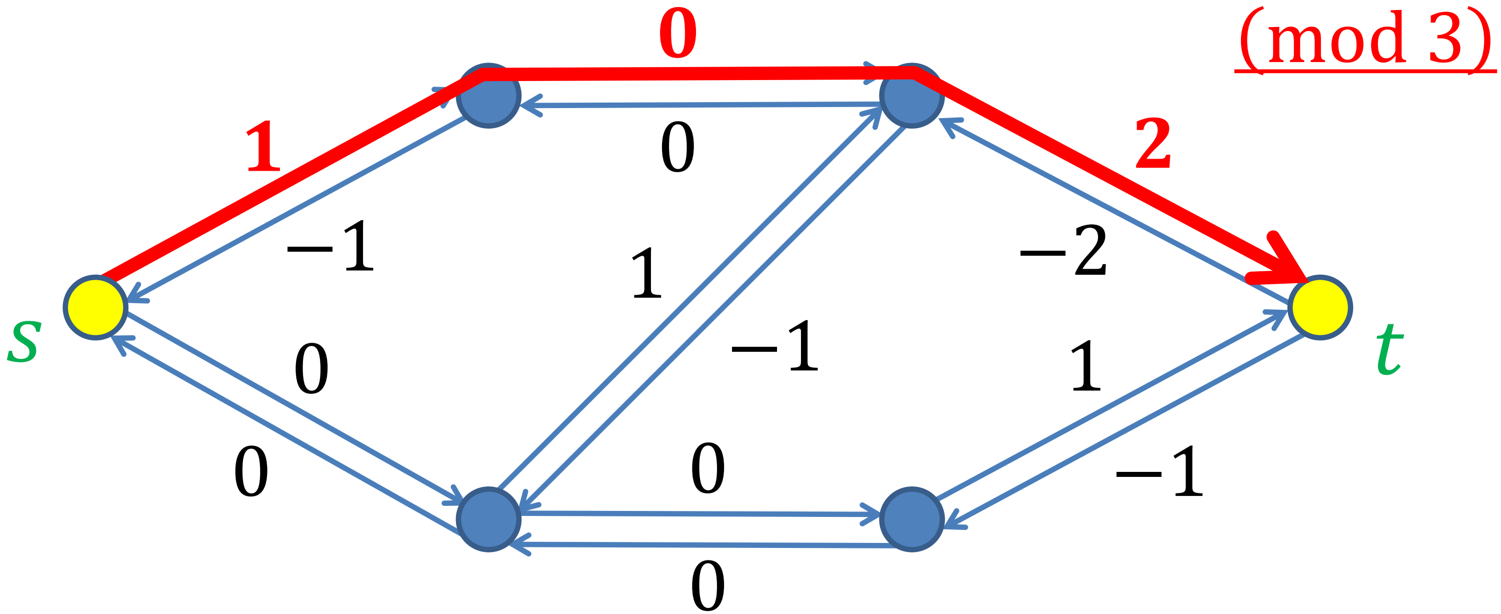
Parity of s–t paths $= \{$Odd$\}$

Possible Labels



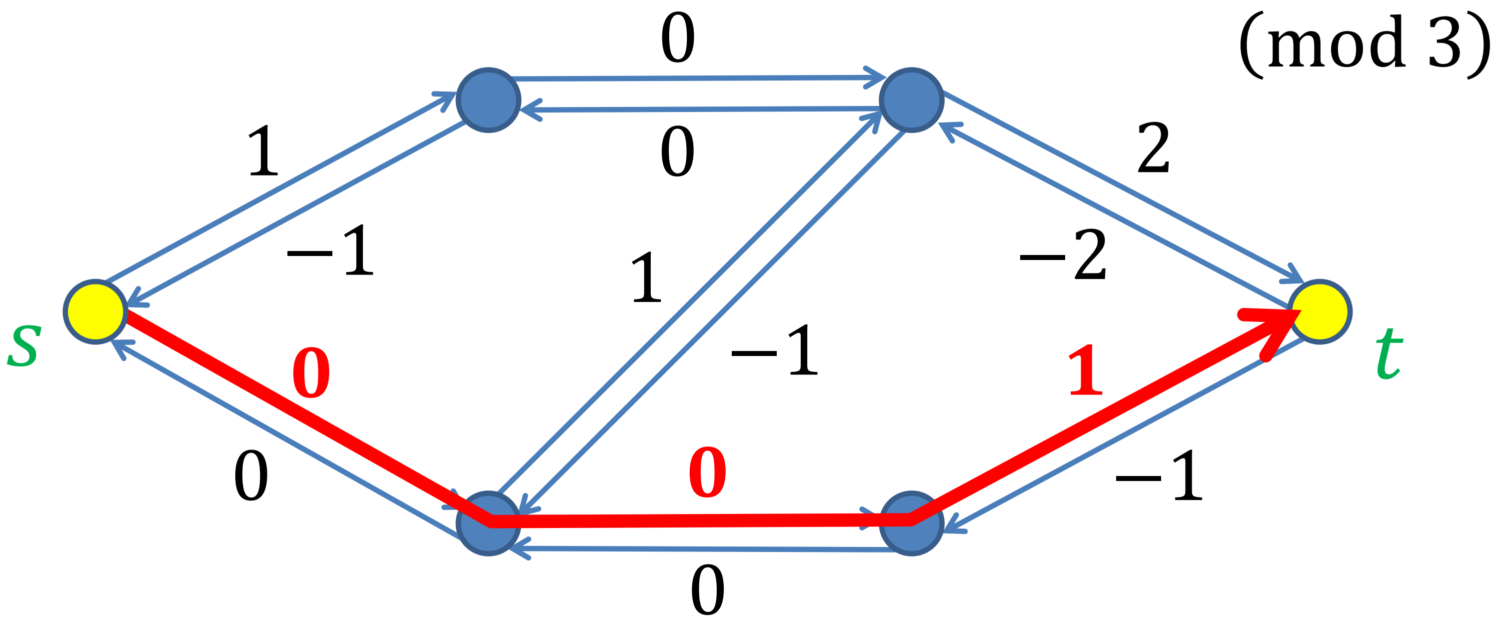
- $0 \;\rightarrow\;$???
- $1 \;\rightarrow\;$???
- $2 \;\rightarrow\;$???

Possible Labels



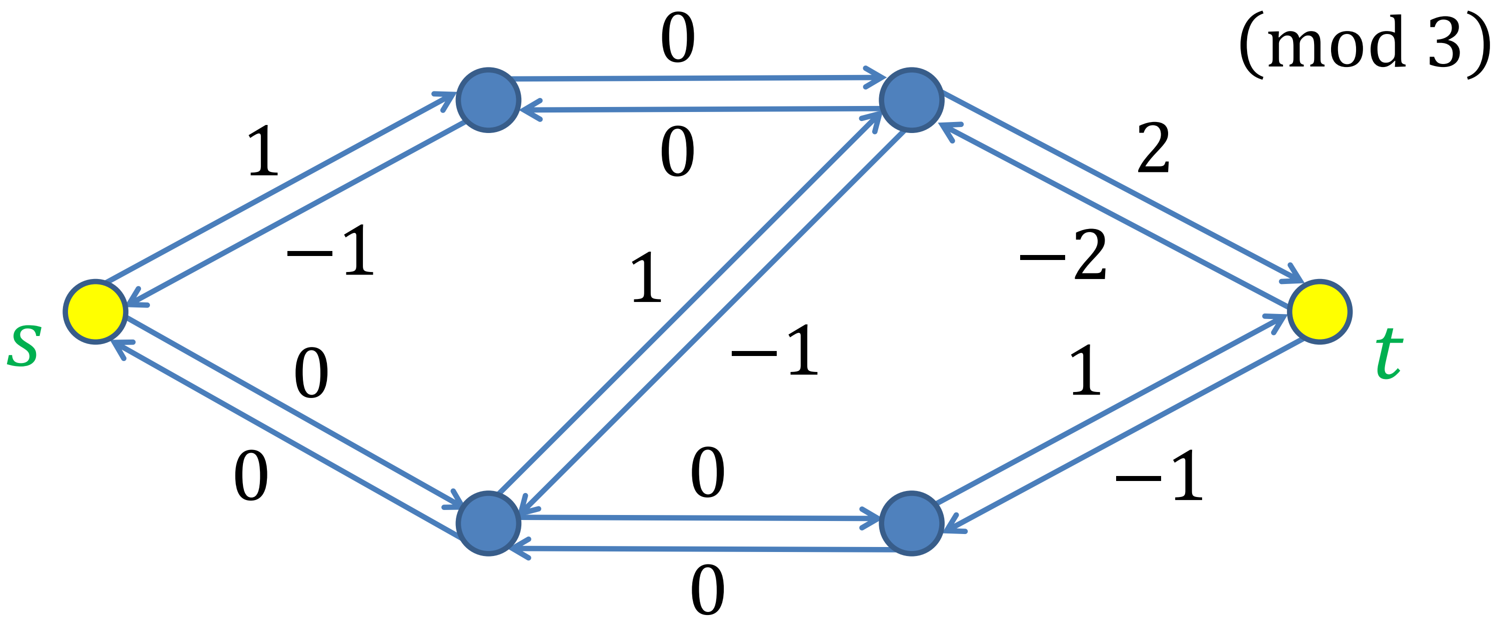
- <u>$0 \rightarrow$ **YES**</u> $\quad 1 + 0 + 2 = 3 \equiv 0$
- $1 \rightarrow$???
- $2 \rightarrow$???

Possible Labels



- $0 \rightarrow$ YES
- <u>$1 \rightarrow$ **YES**</u> $\quad 0 + 0 + 1 = 1$
- $2 \rightarrow$???

Possible Labels

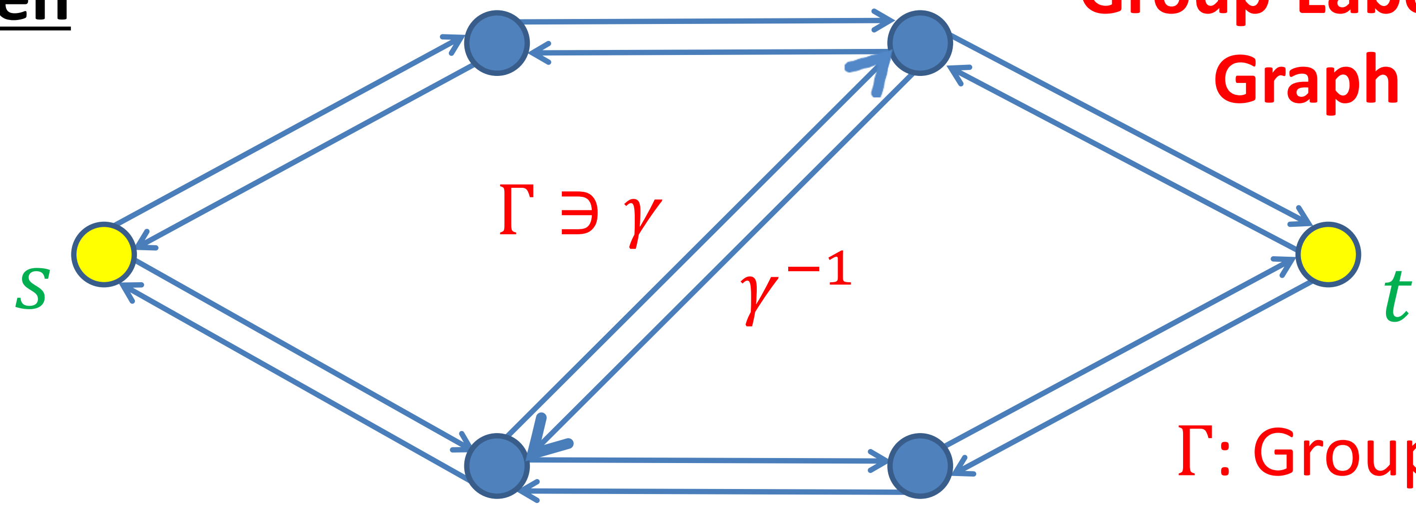


- $0 \rightarrow$ YES
- $1 \rightarrow$ YES
- $2 \rightarrow$ NO

Labels of s–t paths $= \{0, 1\}$

Our Problem

<u>**Given**</u>



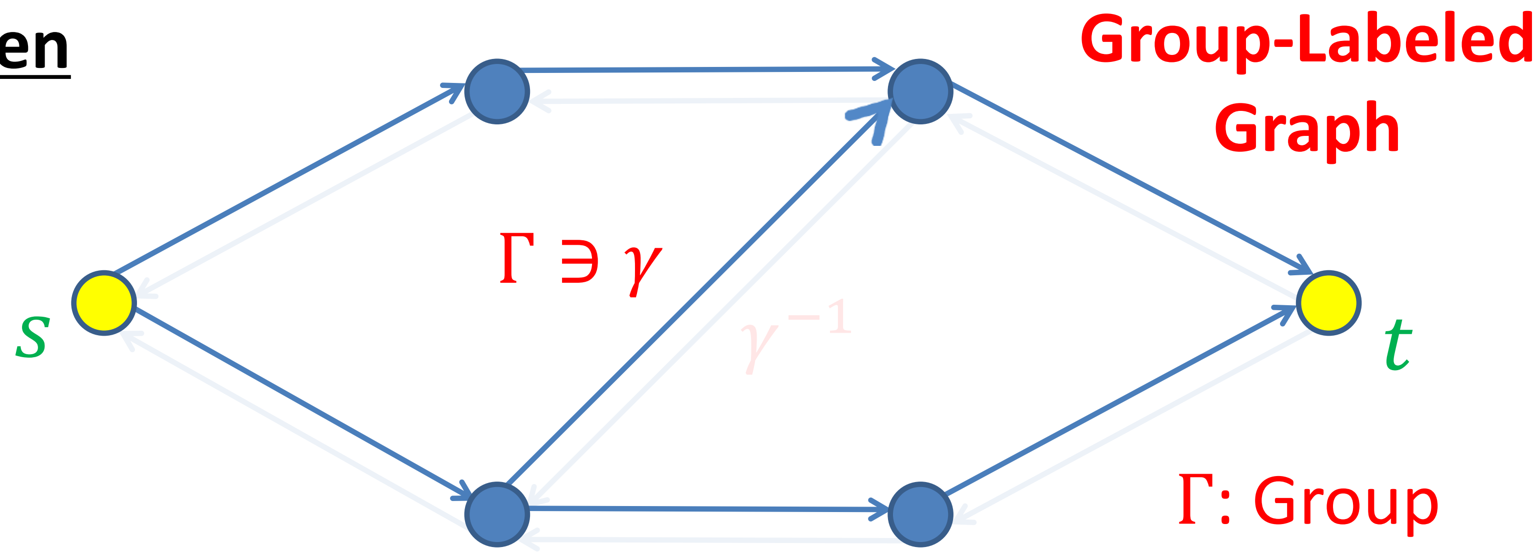
Group-Labeled Graph

Γ: Group

<u>**Find**</u> **Possible Labels of s–t paths**

Our Problem

Given

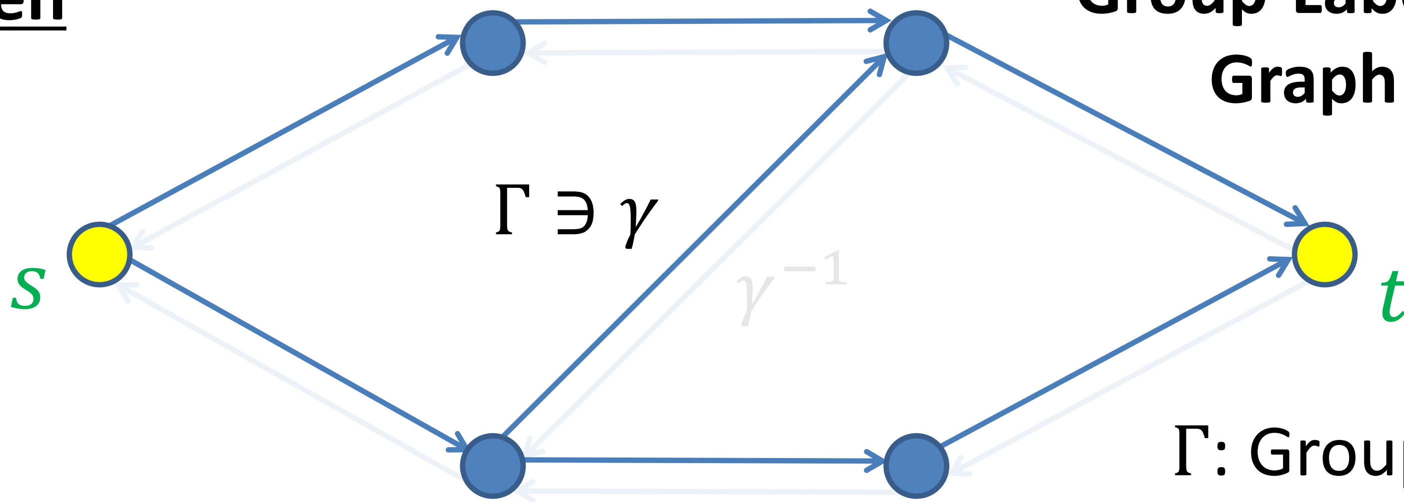


Find **Possible Labels of s–t paths**

Our Problem

<u>Given</u>

Group-Labeled Graph



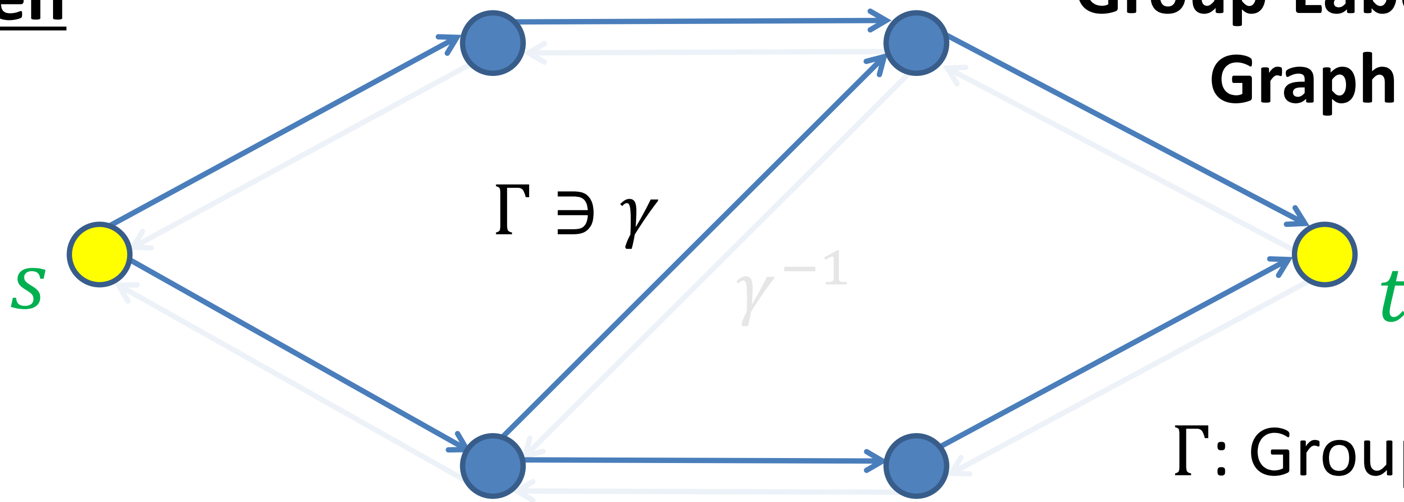
Γ: Group

<u>Find</u> **Possible Labels of s–t paths** $\rightarrow$ **Difficult**

- $l = \{\alpha\}$: Polytime
- $l \supseteq \{\alpha\}$: NP-hard (Hamiltonian Path)
- $l = \{\alpha, \beta\}$: ???

Our Problem

Given



Group-Labeled Graph

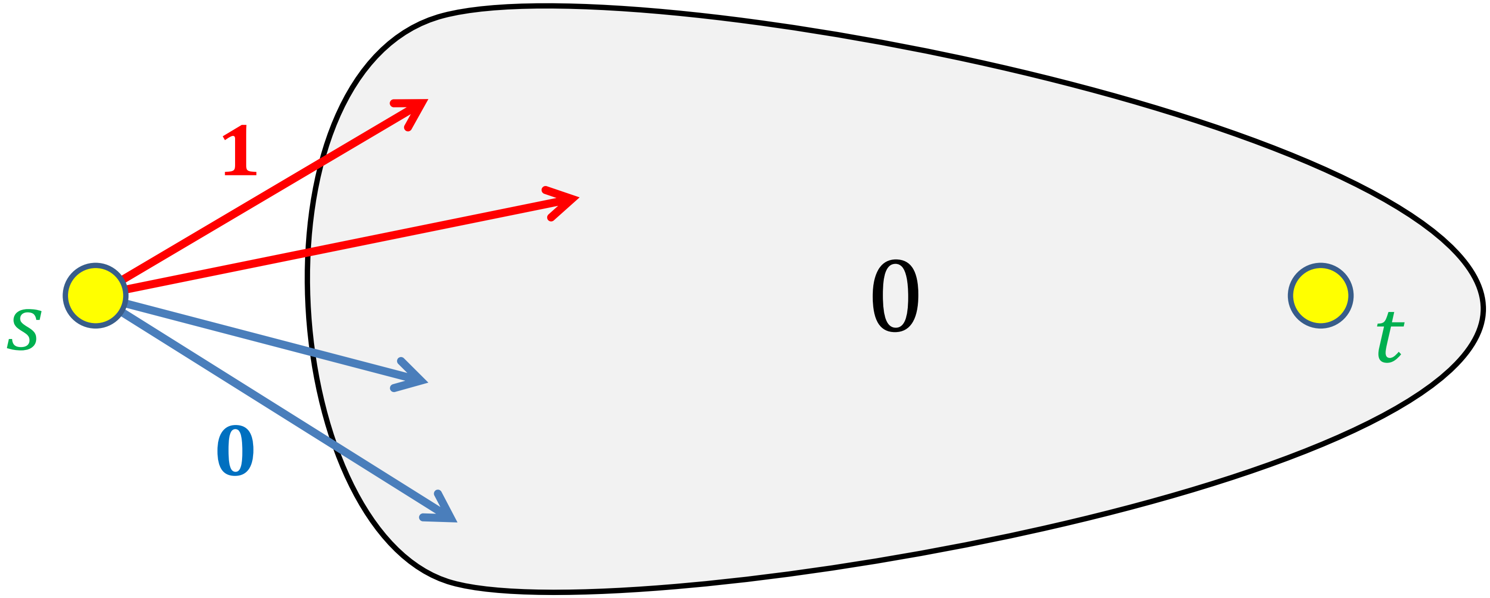
Γ: Group

Find **Possible Labels of s–t paths** → Difficult

- $l = \{\alpha\}$: Polytime
- $l \supseteq \{\alpha\}$: NP-hard (Hamiltonian Path)
- $l = \{\alpha, \beta\}$: **Polytime!!**

Example 1

$\mathbf{Z}$-Labeled Graph
$\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$



$l = \{0, 1\}$

Example 1

$\mathbf{Z}$-Labeled Graph

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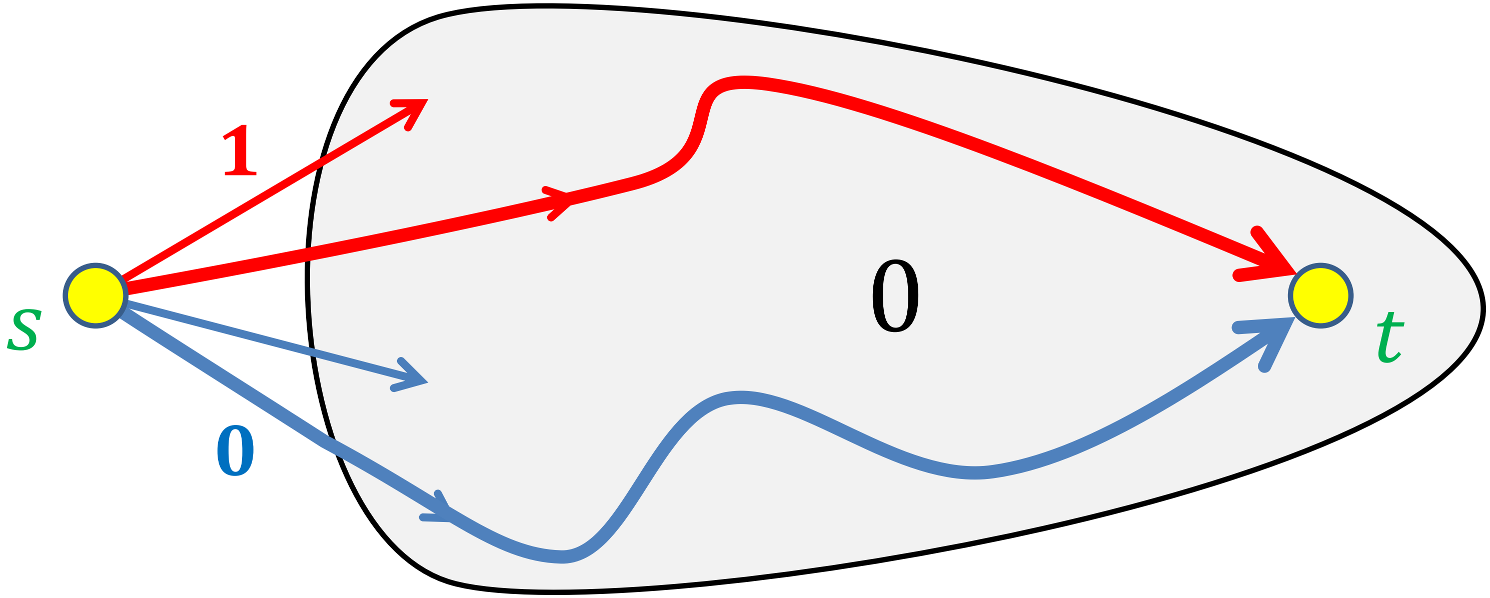

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Example 2

$\mathbf{Z}$-Labeled Graph
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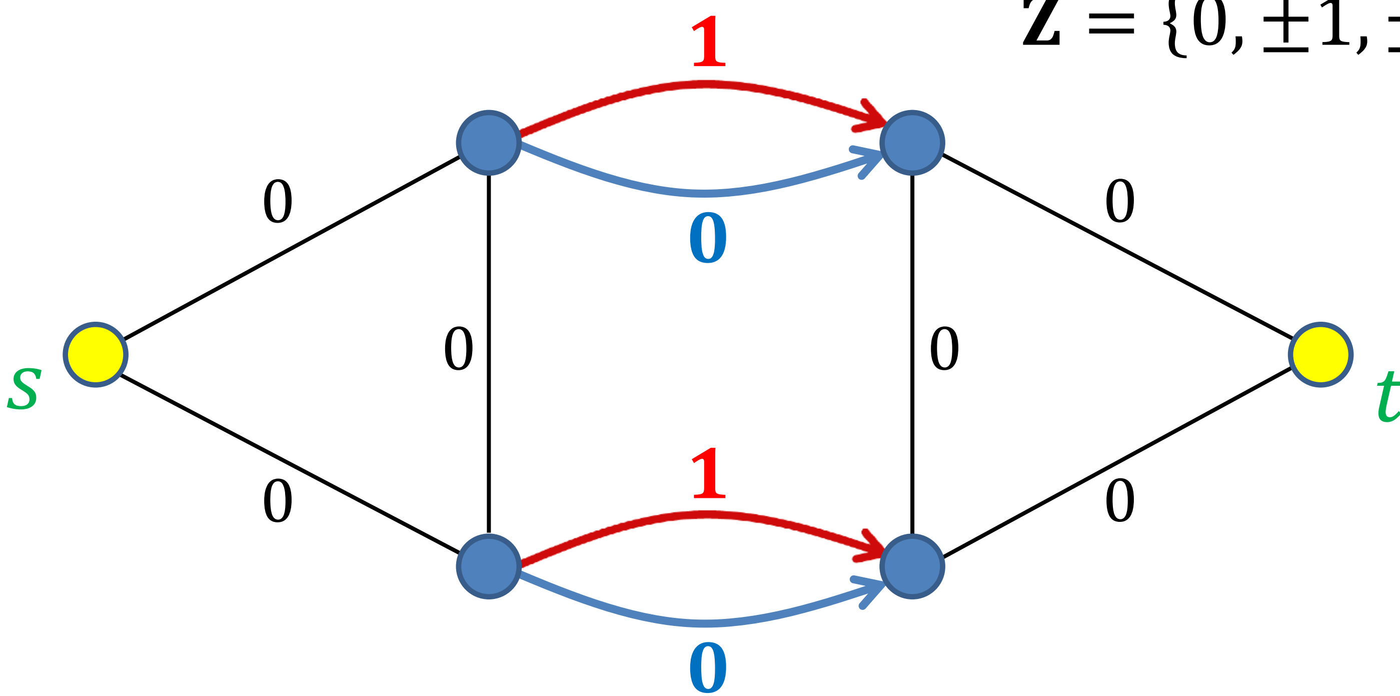

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Example 2

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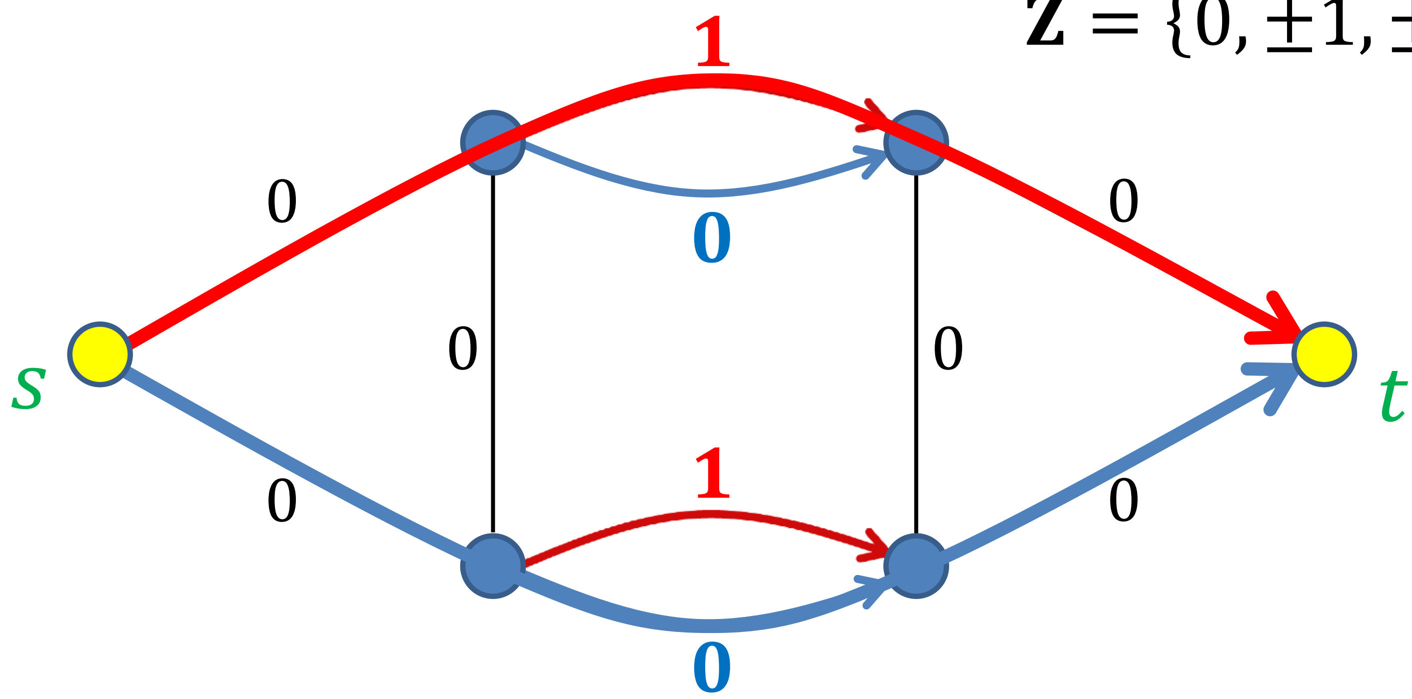

$l = \{0, 1\}$

Example 3

Z-Labeled Graph

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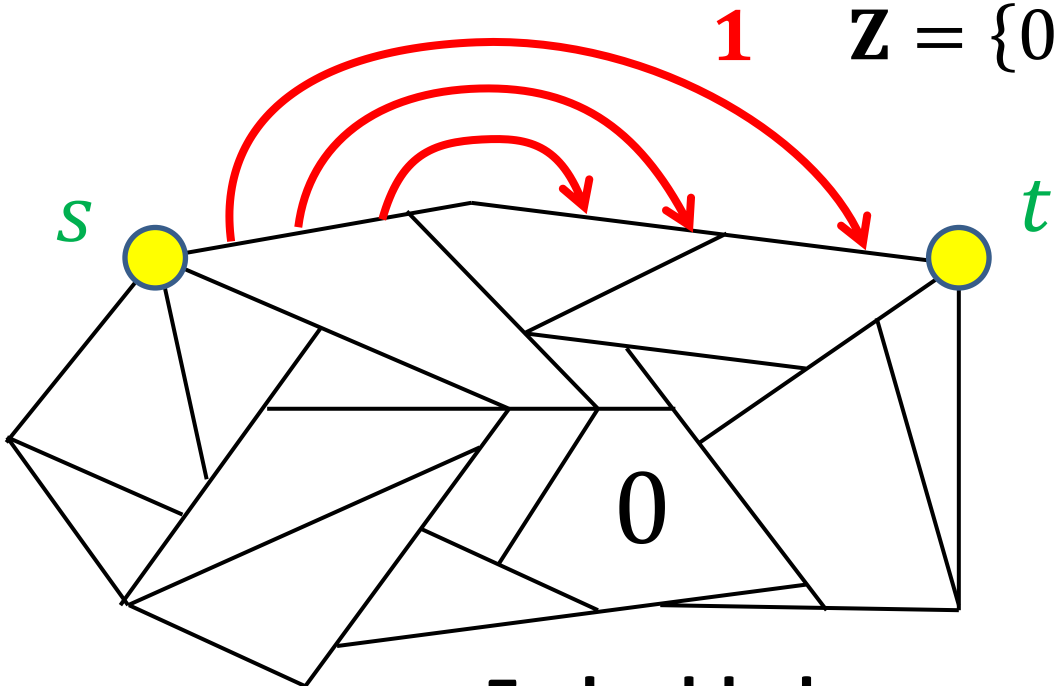

Embedded

$l = \{0, 1\}$

Example 3

$\mathbf{Z}$-Labeled Graph

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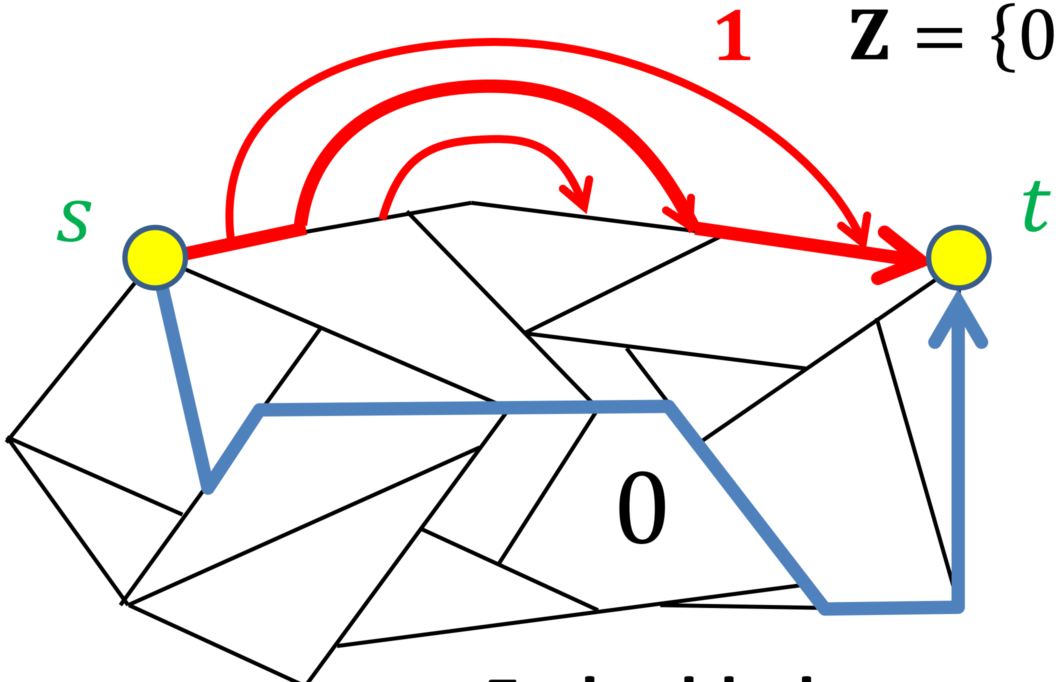

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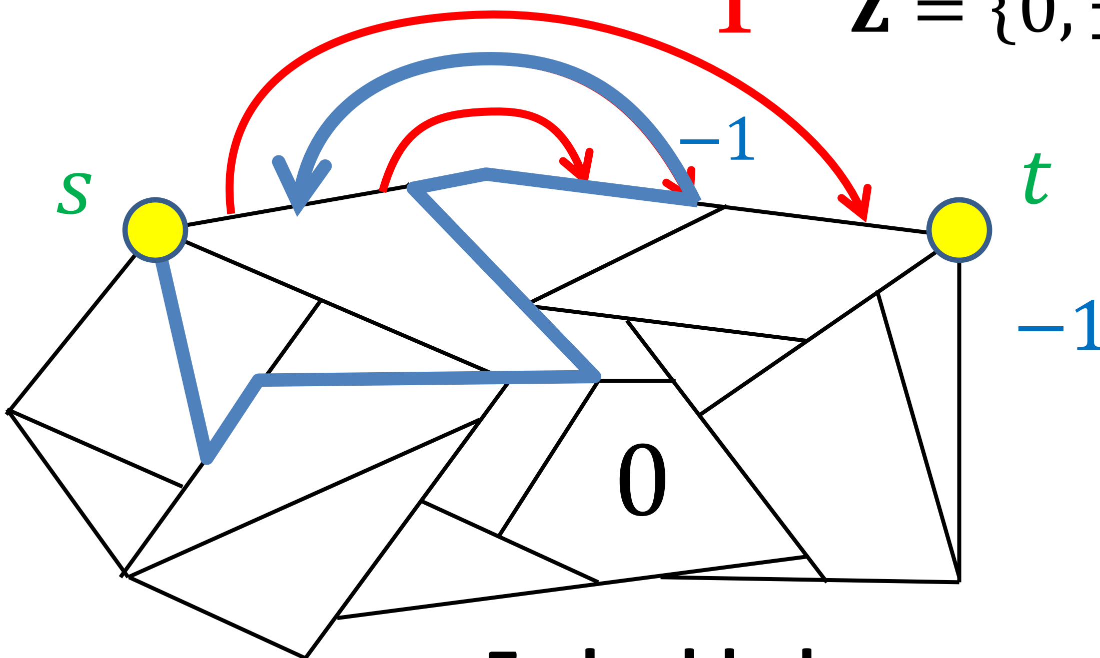

$l = \{0, 1\}$

Example 3

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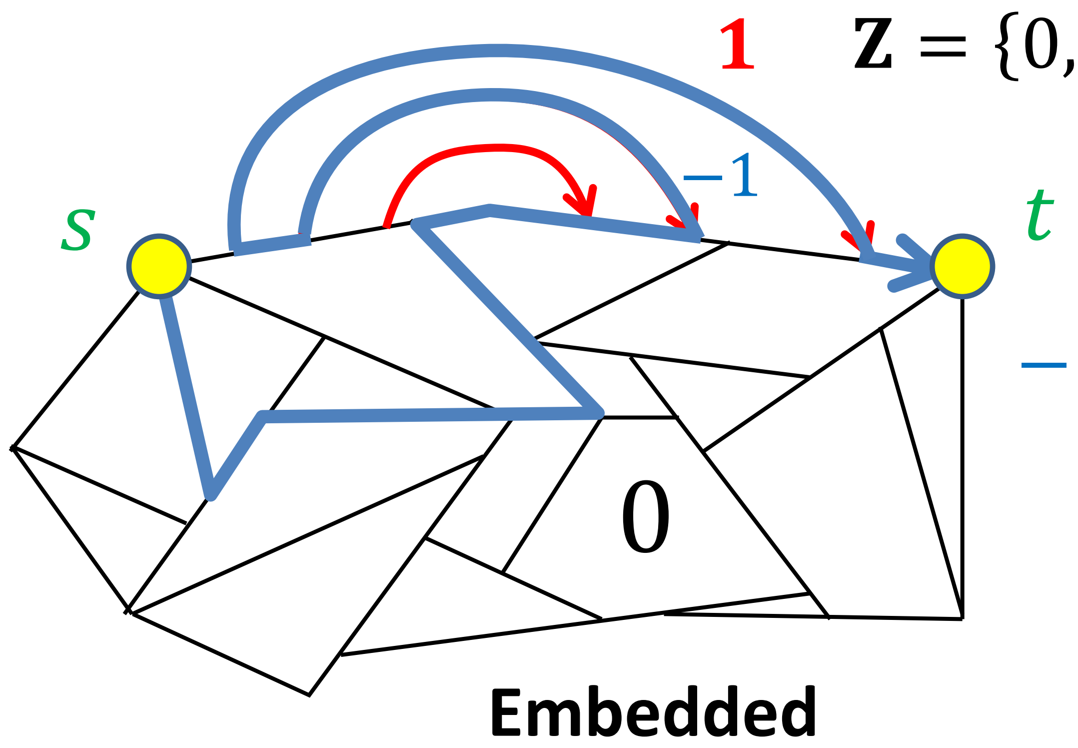


$-1 + 1 = 0$

$l = \{0, 1\}$

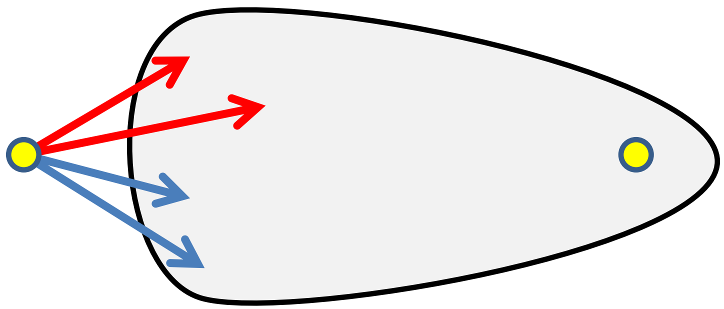
Our Result (Characterization)

Thm. $l = \{\alpha, \beta\}$

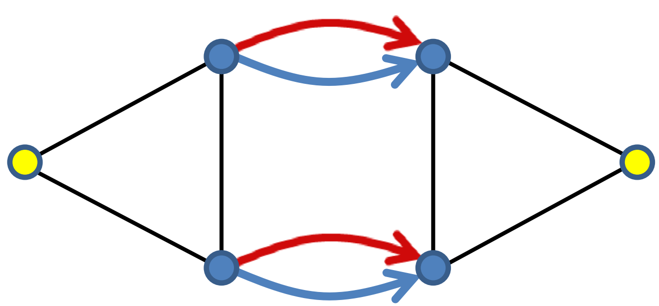
$\Updownarrow$

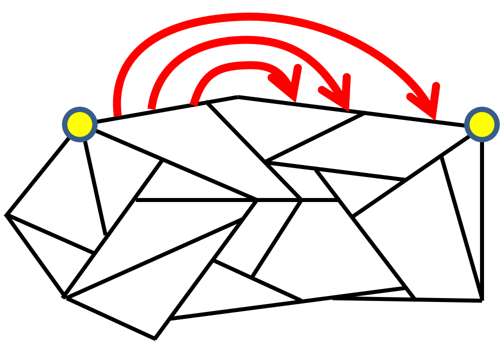
Reducible to one of them

[K.–K.–Y. 2015]



Essential





Our Result (Characterization)

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$$\Updownarrow$$

Reducible to one of them

[K.–K.–Y. 2015]

Polytime Testable

- **NOT** Depends on Group
- Assume **Constant-time Group Operations**
  (e.g. Addition, Comparison, …)

Overview

Finding an s–t path

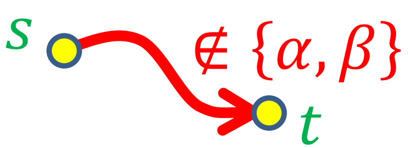
with 2 Labels Forbidden

Characterization for **2 Possible Labels**

[K.–K.–Y. 2015]

$\underline{l = \{\alpha, \beta\}}$

Polytime



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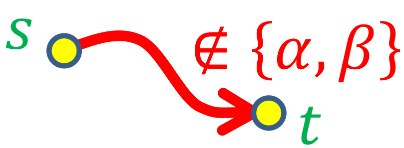
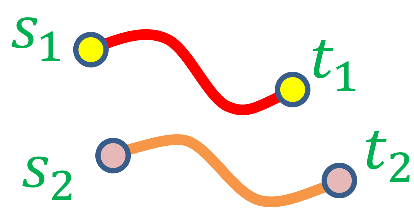


Generalized



Generalized

2-disjoint Paths Problem

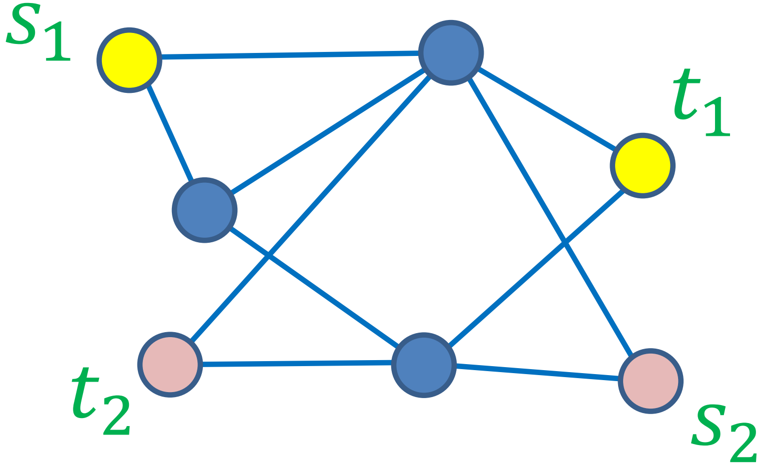
Polytime

Characterization for **2-disjoint Paths**

[Seymour 1980, Shiloach 1980, Thomassen 1980]

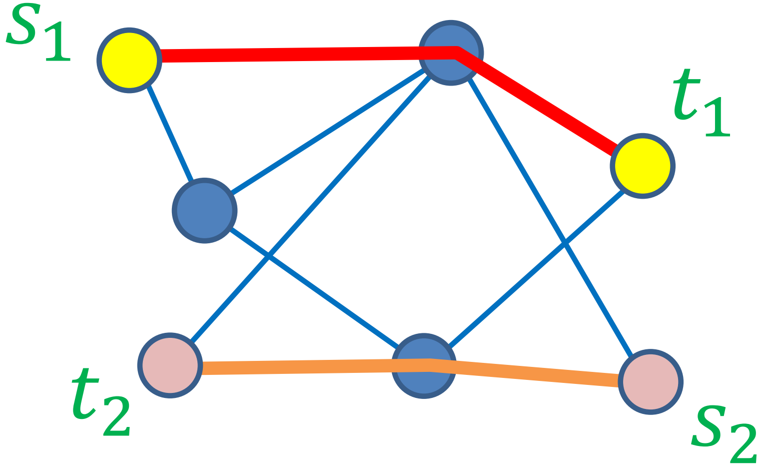
Generalizing 2-disjoint Paths

Undirected Graph



Generalizing 2-disjoint Paths

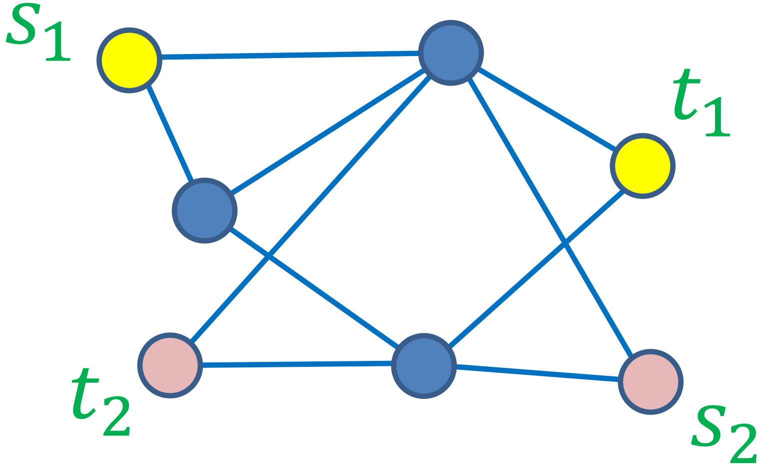
Undirected Graph



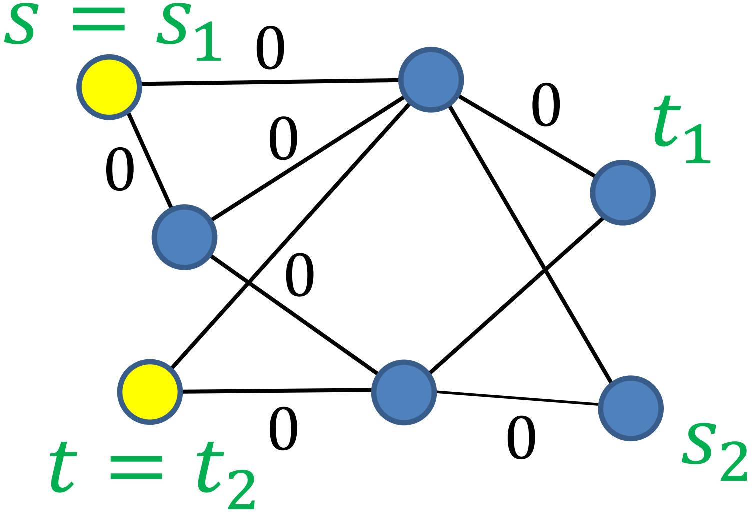
2-disjoint paths

Generalizing 2-disjoint Paths

Undirected Graph

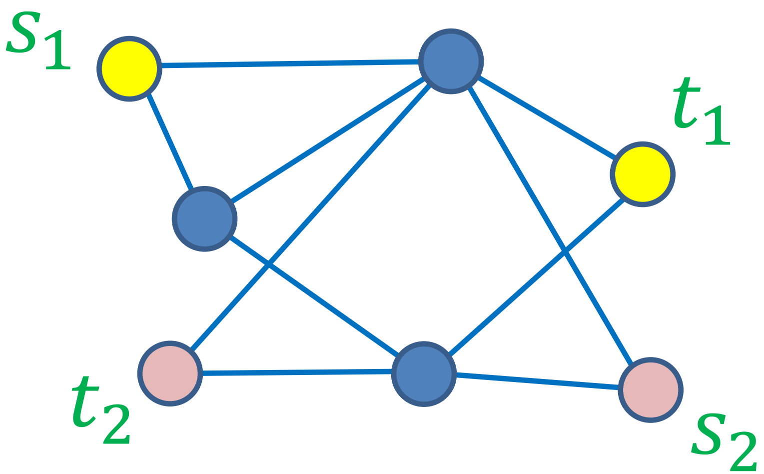


$\mathbf{Z}_3$-Labeled Graph
$\mathbf{Z}_3 = \{-1, 0, 1\}$



Generalizing 2-disjoint Paths

Undirected Graph



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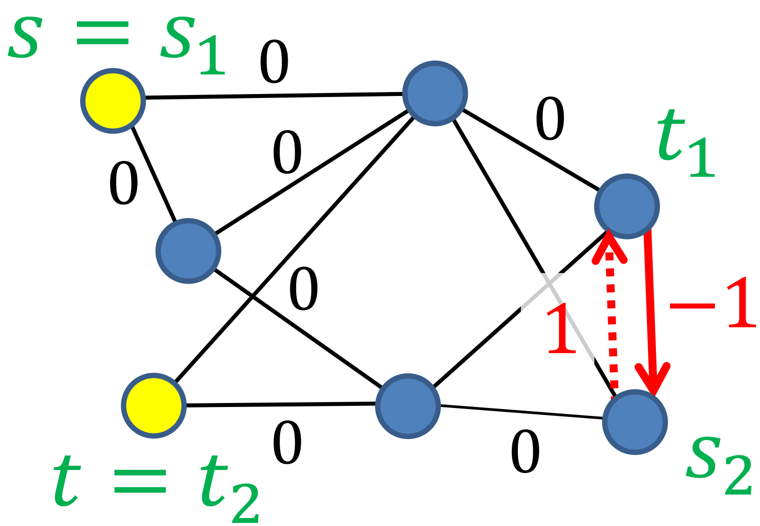

Generalizing 2-disjoint Paths

Undirected Graph

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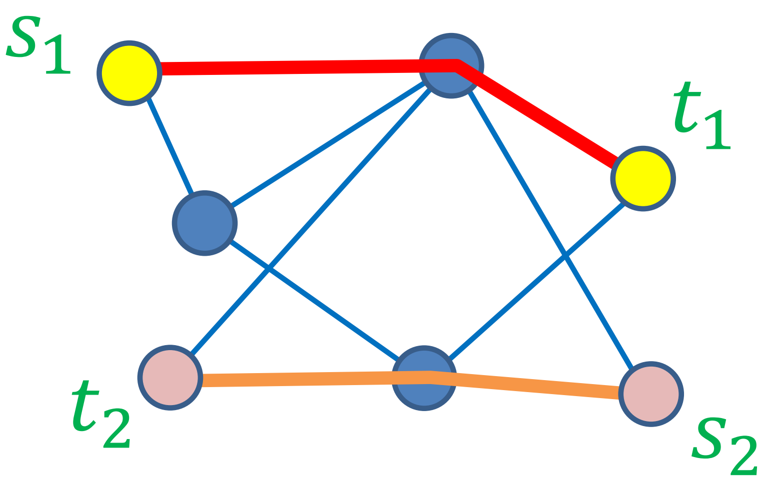

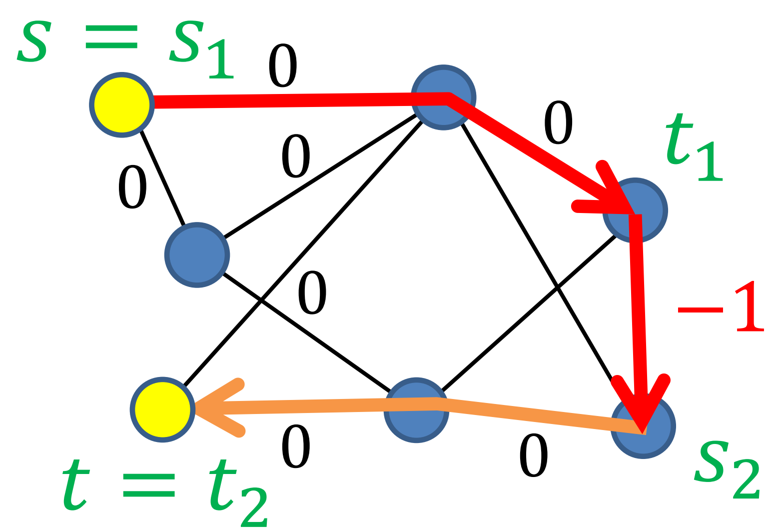

2-disjoint paths $\Leftrightarrow$ Label -1

Generalizing 2-disjoint Paths

Undirected Graph

$\mathbf{Z}_3$-Labeled Graph
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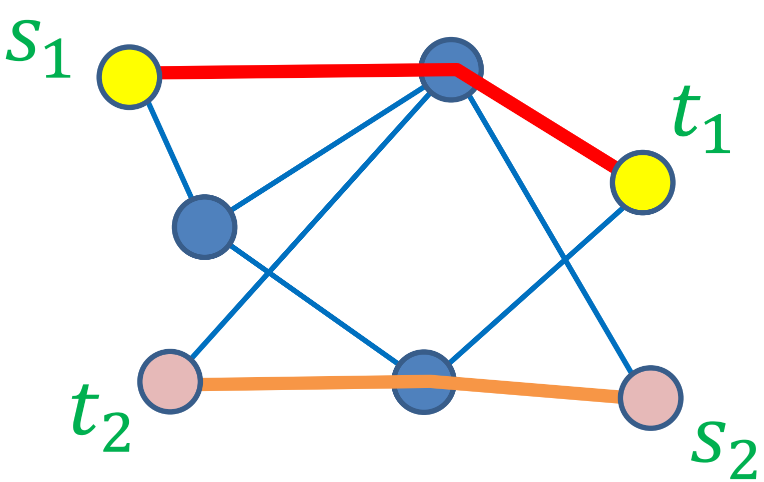
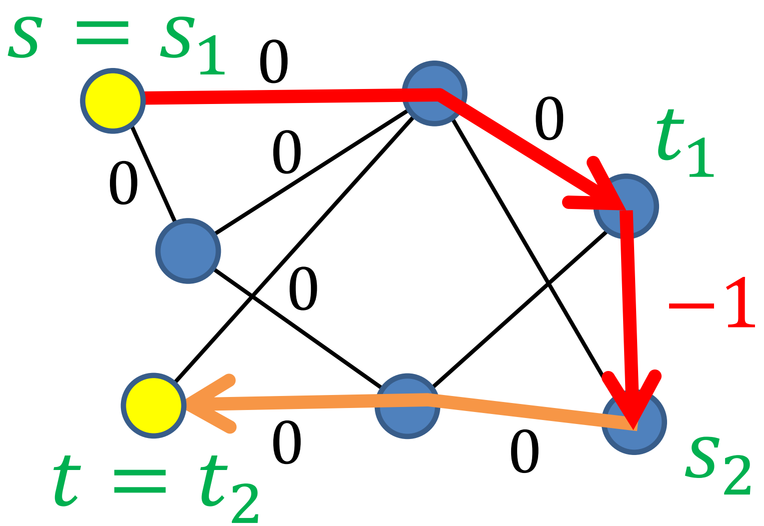




2-disjoint paths $\Leftrightarrow$ Label -1

$\Leftrightarrow$ **Neither 0 nor 1**

Overview

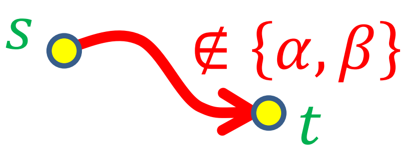
Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

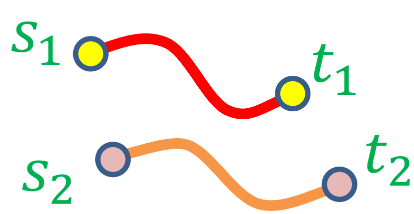
Polytime

[K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$



Generalized



Generalized

2-**disjoint Paths Problem**

Characterization
for
2-**disjoint Paths**

Polytime

[Seymour 1980,
Shiloach 1980,
Thomassen 1980]

Overview

Finding an s–t path
with 2 Labels Forbidden

Polytime

[K.–K.–Y. 2015]

Characterization
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2 Possible Labels

$l = \{\alpha, \beta\}$

Generalized

Generalized

2-**disjoint Paths Problem**

Polytime

[Seymour 1980,
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Characterization
for
2-**disjoint Paths**

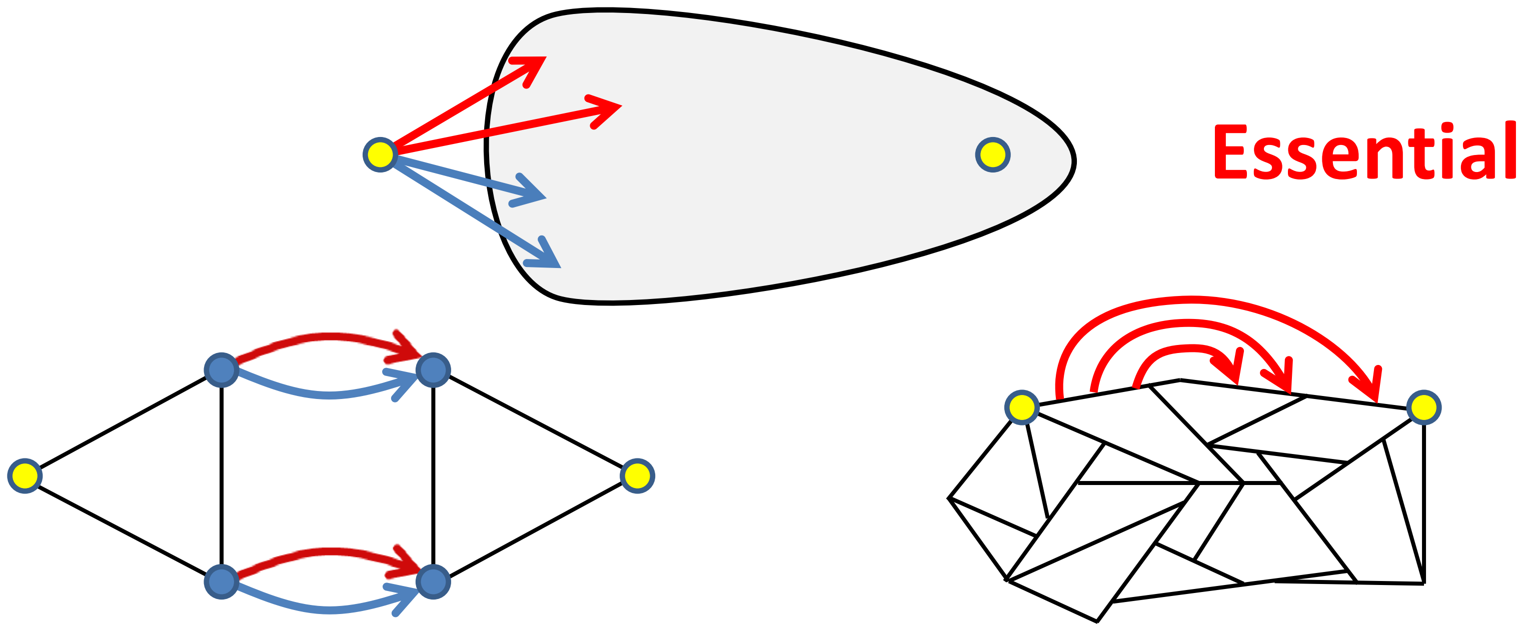
Our Result (Characterization)

Thm. $l = \{\alpha, \beta\}$

$\Updownarrow$

Reducible to one of them

[K.–K.–Y. 2015]



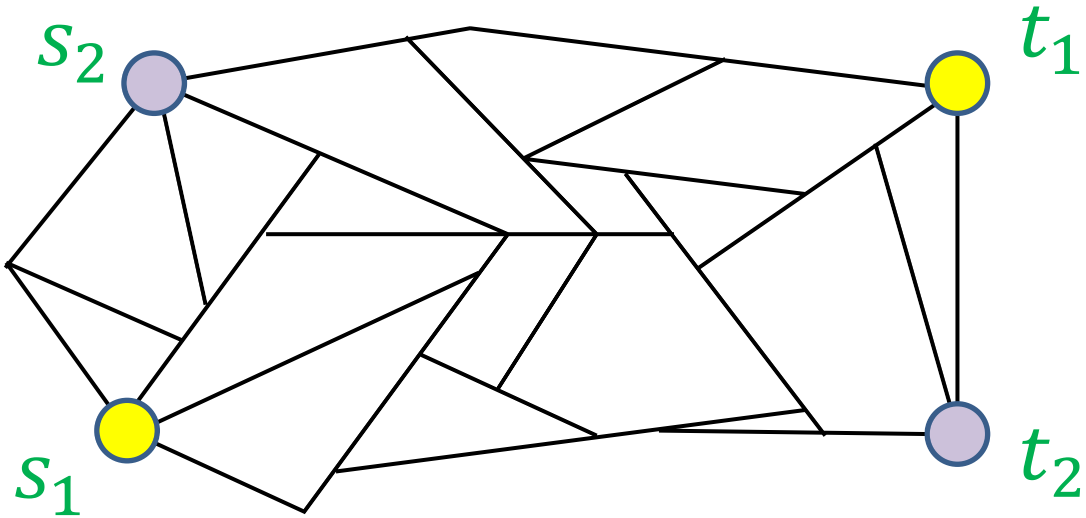
Characterization for 2-disjoint Paths

<u>Thm.</u> <u>**NO**</u> disjoint s_1–t_1, s_2–t_2 paths

$$\Updownarrow$$

Reducible so that <u>**Planar Embeddable**</u>

[Seymour 1980]



Essential

Characterization for 2-disjoint Paths

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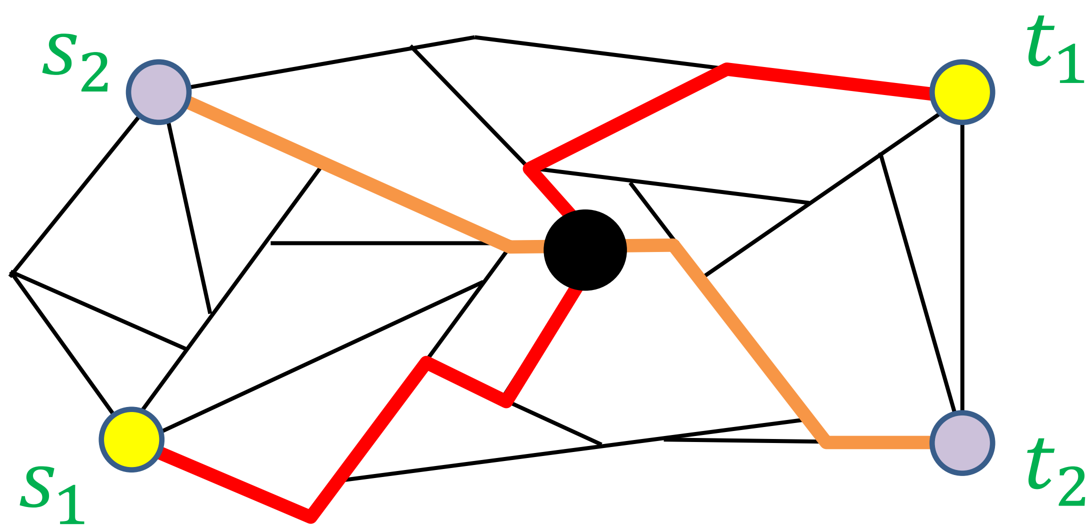
Reducible so that **Planar Embeddable**

[Seymour 1980]



Essential

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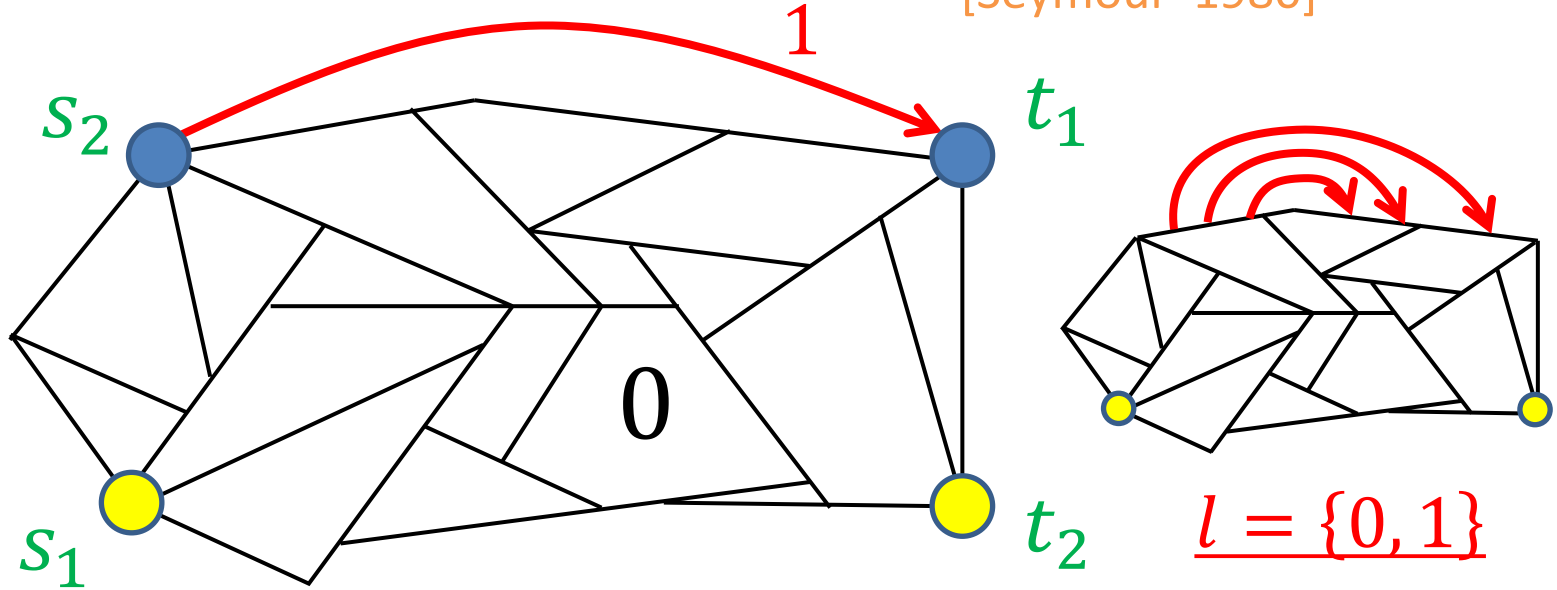
[Seymour 1980]



Characterization for 2-disjoint Paths

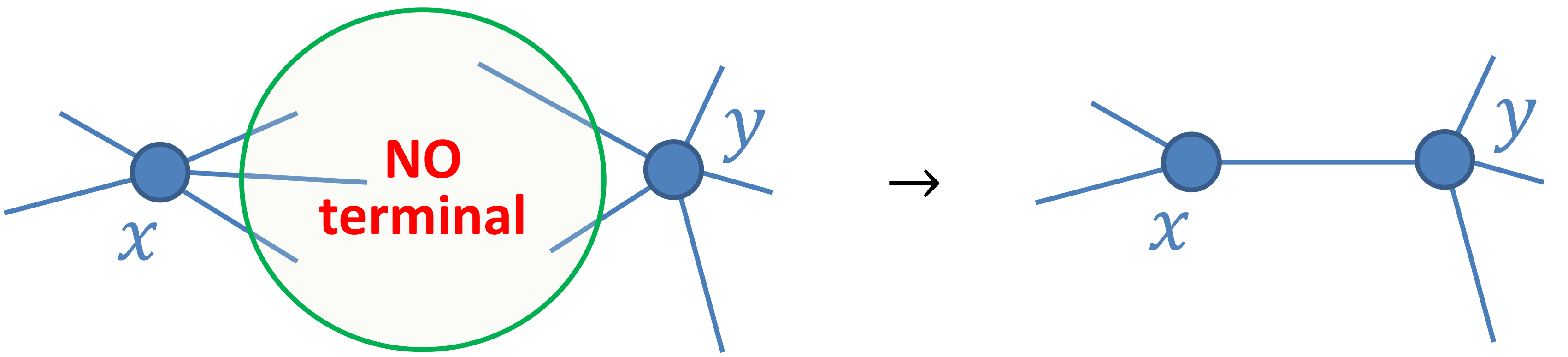
Thm. **NO** disjoint $s_1 – t_1$, $s_2 – t_2$ paths

$$\Updownarrow$$

Reducible so that **Planar Embeddable**

[Seymour 1980]

Contraction of 2-cut



Characterization for 2-disjoint Paths

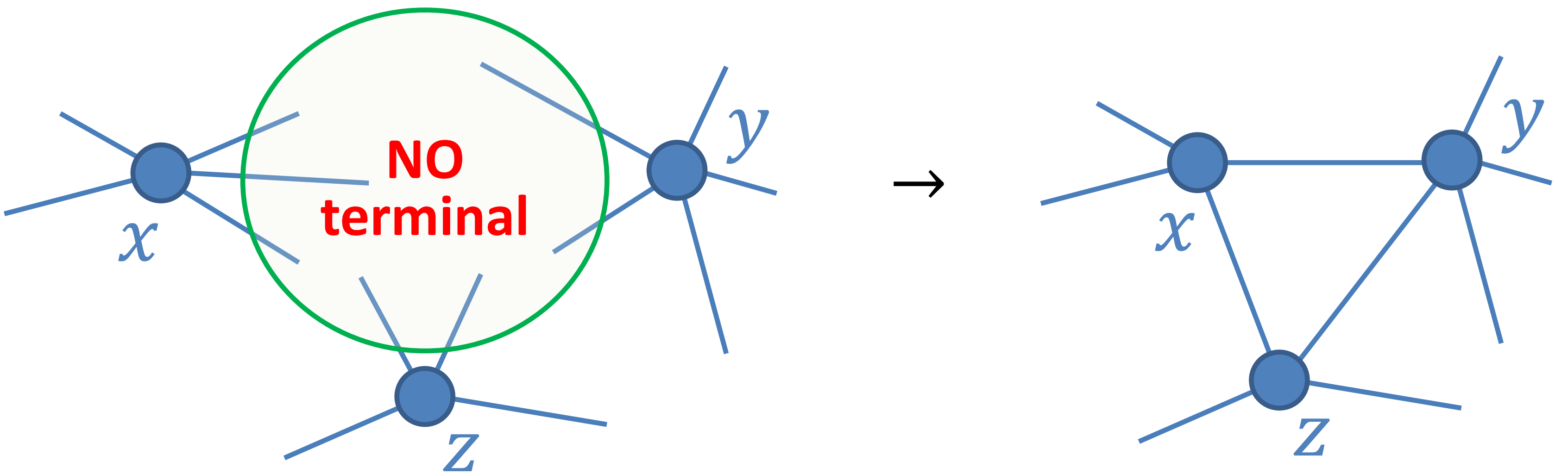
Thm. **NO** disjoint s_1–t_1, s_2–t_2 paths

$$\Updownarrow$$

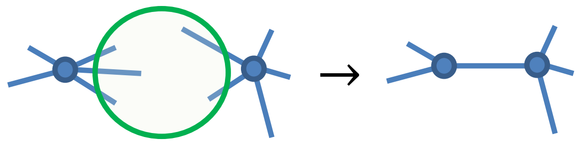
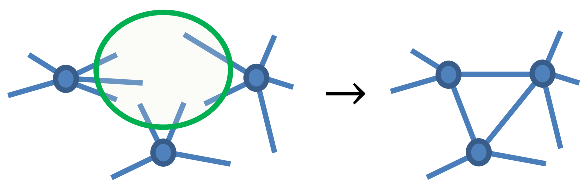
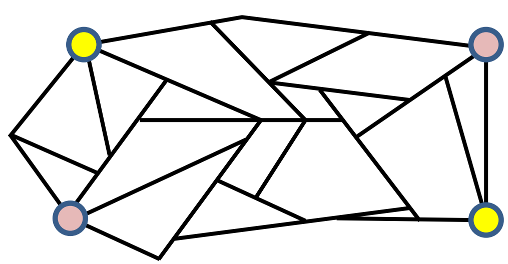
Reducible so that **Planar Embeddable**

[Seymour 1980]

Contraction of 3-cut



Contrast between Two Characterizations

	<u>NO</u> 2-**disjoint Paths** [Seymour 1980]	**Exactly 2 Possible Labels** [K.–K.–Y. 2015]
Reducing Operations	Contraction of 2-cut 	
	Contraction of 3-cut 	
Essential Cases	Planar Embeddable 	

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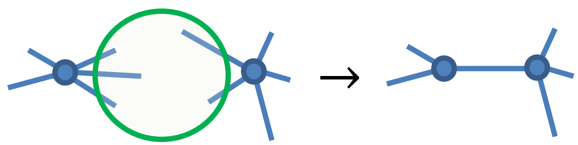
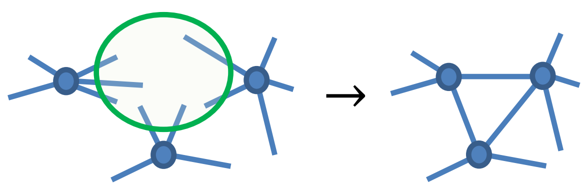
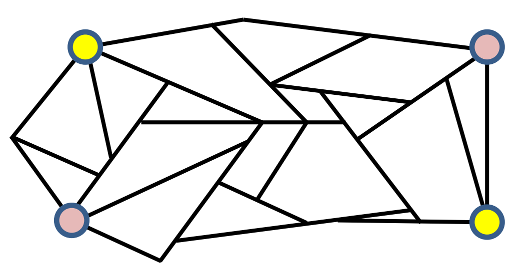
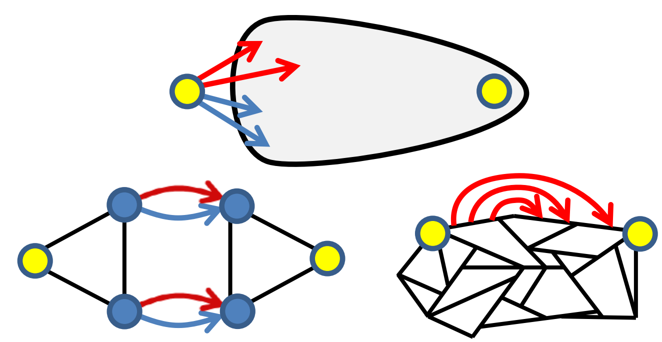
Reducing Operations	Contraction of 2-cut 	
	Contraction of 3-cut 	
Essential Cases	Planar Embeddable 	

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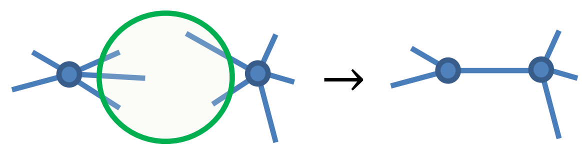
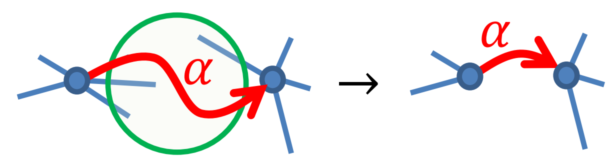
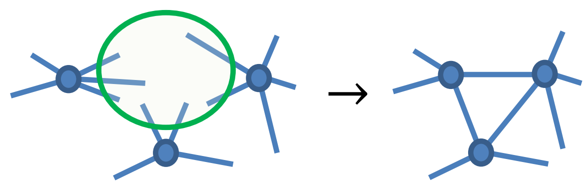
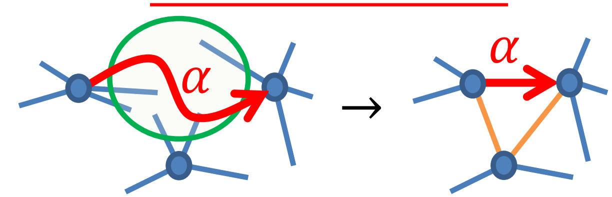
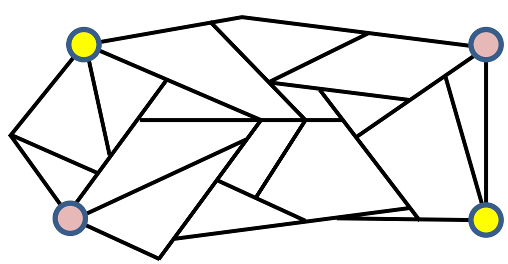
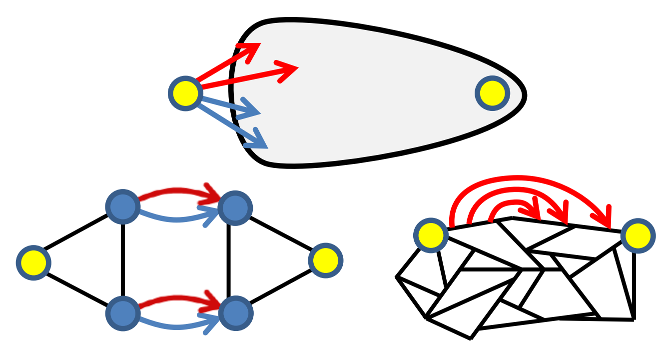
	<u>NO</u> 2-disjoint Paths [Seymour 1980]	**Exactly 2 Possible Labels** [K.–K.–Y. 2015]
Reducing Operations	Contraction of 2-cut 	**<u>2-contraction</u>** 
	Contraction of 3-cut 	**<u>3-contraction</u>** 
Essential Cases	Planar Embeddable 	

Our Result (Algorithm)

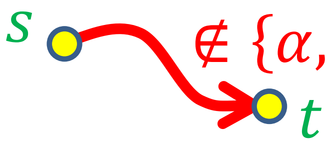
Finding an s–t path

with 2 Labels Forbidden

Characterization

for

2 Possible Labels



$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

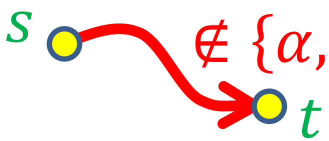
$l = \{\alpha, \beta\}$

Our Result (Algorithm)

Finding an s–t path

with 2 Labels Forbidden

Characterization for **2 Possible Labels**



$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

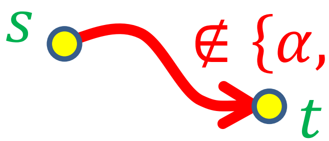
- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)

cf. $l = \{\alpha\}$ is Easy

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels



$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ $\rightarrow$ **Certification for "NO"**

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ $\rightarrow$ **Certification for "NO"**
- $l \nsubseteq \{\alpha, \beta\}$ $\rightarrow$ $\exists \gamma \in l \setminus \{\alpha, \beta\}$

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

Characterization
for
2 Possible Labels

$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\}$ $\rightarrow$ **Certification for “NO”**
- $l \nsubseteq \{\alpha, \beta\}$ $\rightarrow$ $\exists \gamma \in l \setminus \{\alpha, \beta\}$
  - $|l| \leq 2$ $\rightarrow$ Paths of **ALL** **Possible Labels**
  - $|l| \geq 3$ $\rightarrow$ Paths of **3 Distinct** **Labels**

Our Result (Algorithm)

Finding an s–t path
with 2 Labels Forbidden

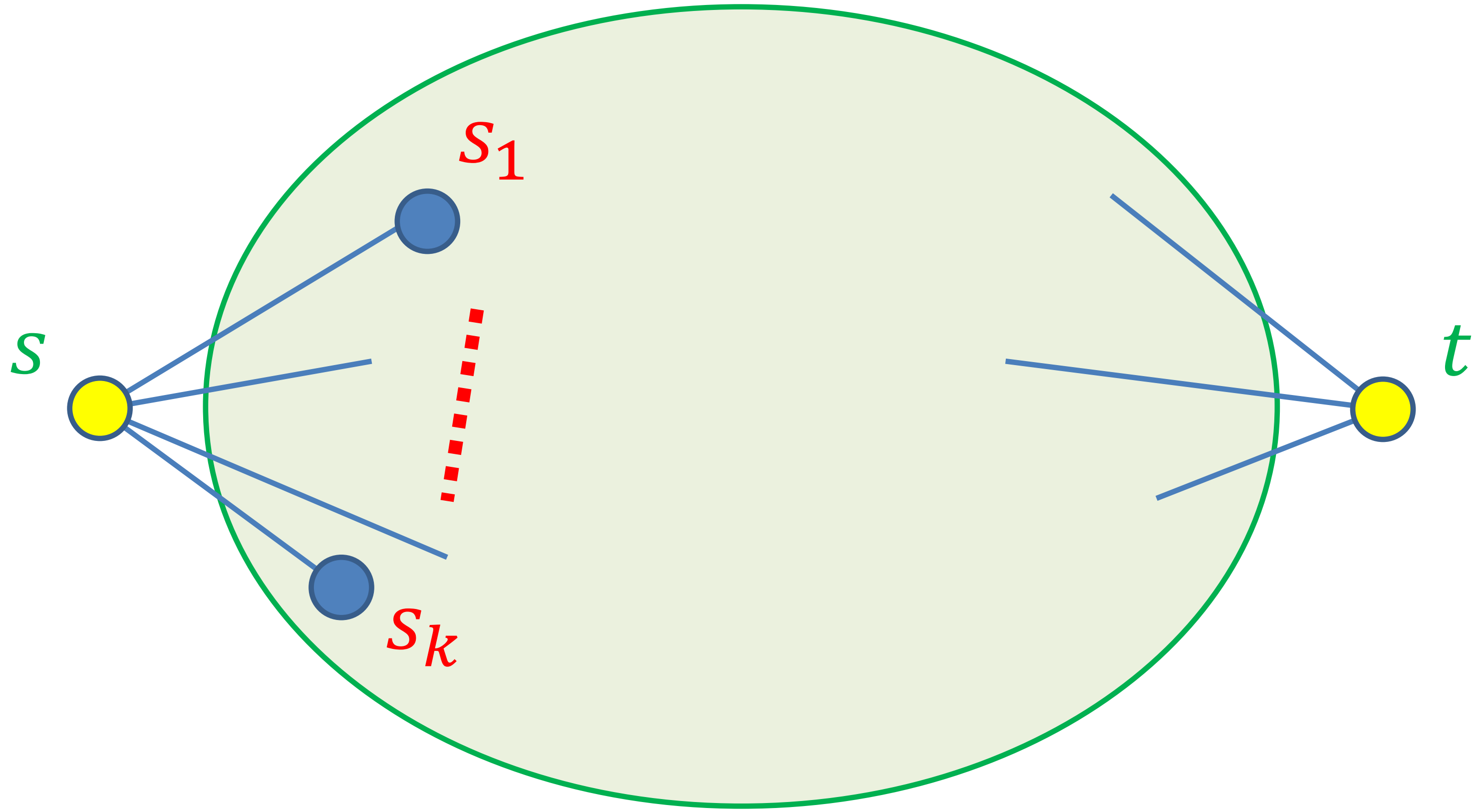
Characterization
for
2 Possible Labels

$\notin \{\alpha, \beta\}$ **Polytime** [K.–K.–Y. 2015]

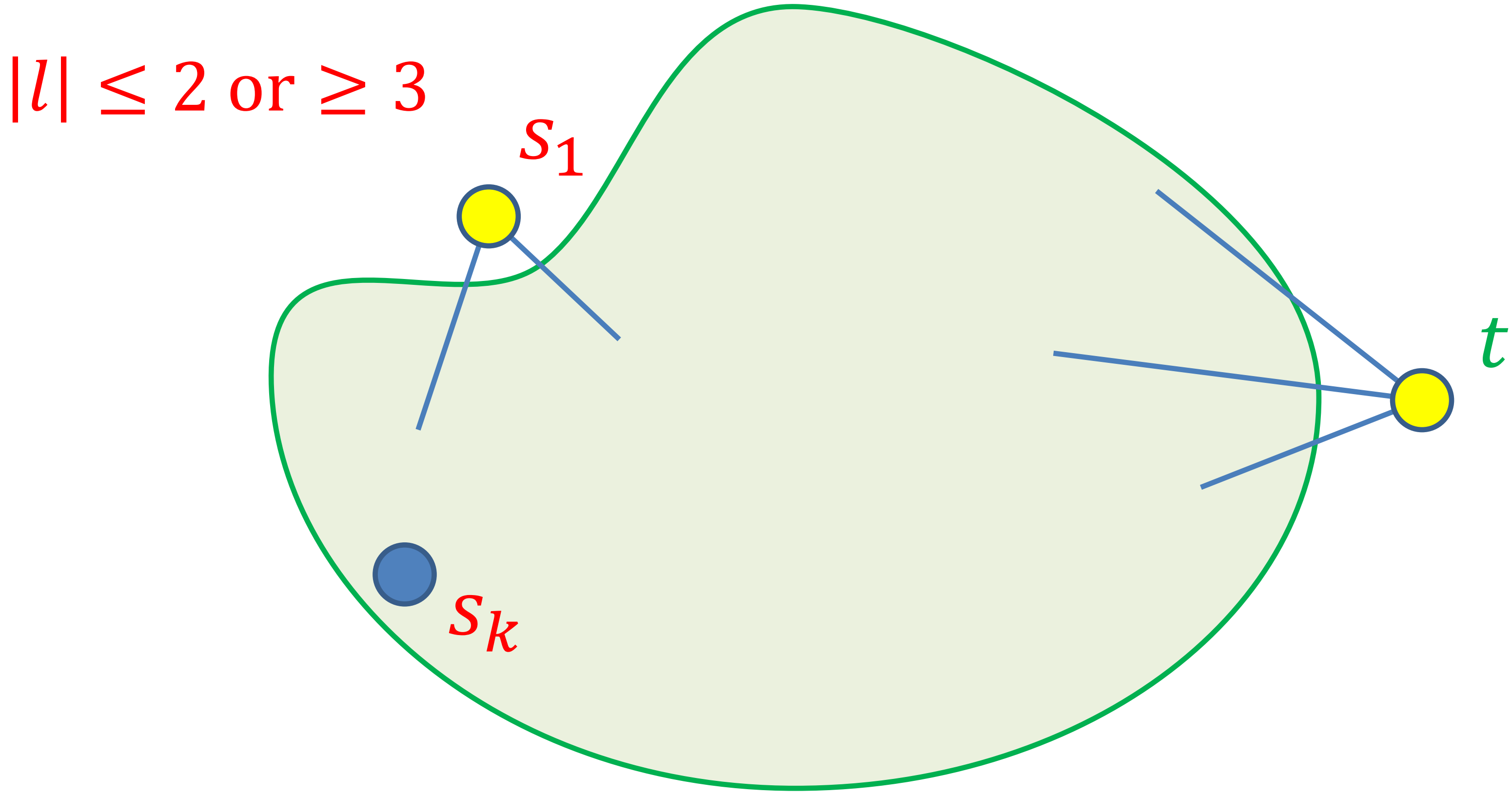
$l = \{\alpha, \beta\}$

- Test "$l \subseteq \{\alpha, \beta\}$ or NOT" (**Based on Our Char.**)
- $l \subseteq \{\alpha, \beta\} \rightarrow$ **Certification for "NO"**
- $l \nsubseteq \{\alpha, \beta\} \rightarrow \exists \gamma \in l \setminus \{\alpha, \beta\}$
  - $|l| \leq 2 \rightarrow$ Paths of **ALL Possible Labels**
  - $|l| \geq 3 \rightarrow$ Paths of **3 Distinct Labels**

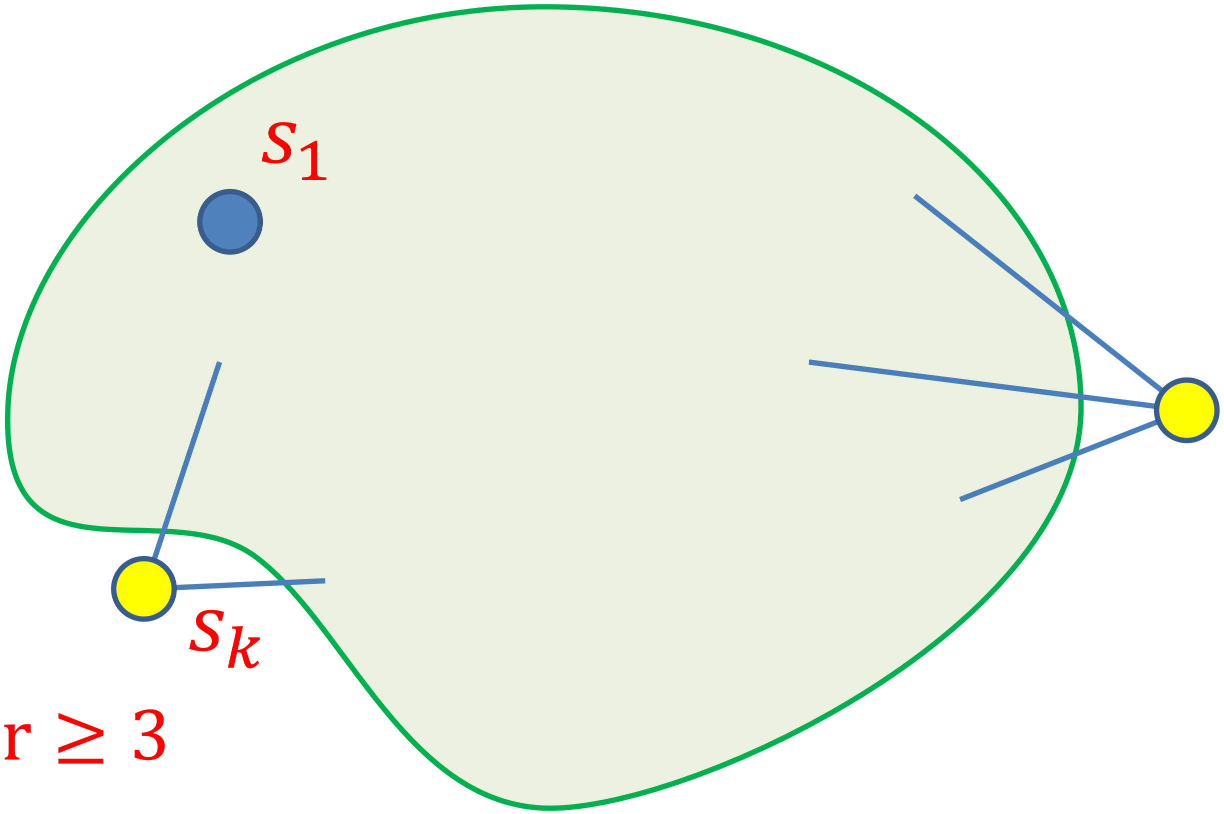
Finding s–t Paths of 3 Distinct Labels



Finding s–t Paths of 3 Distinct Labels

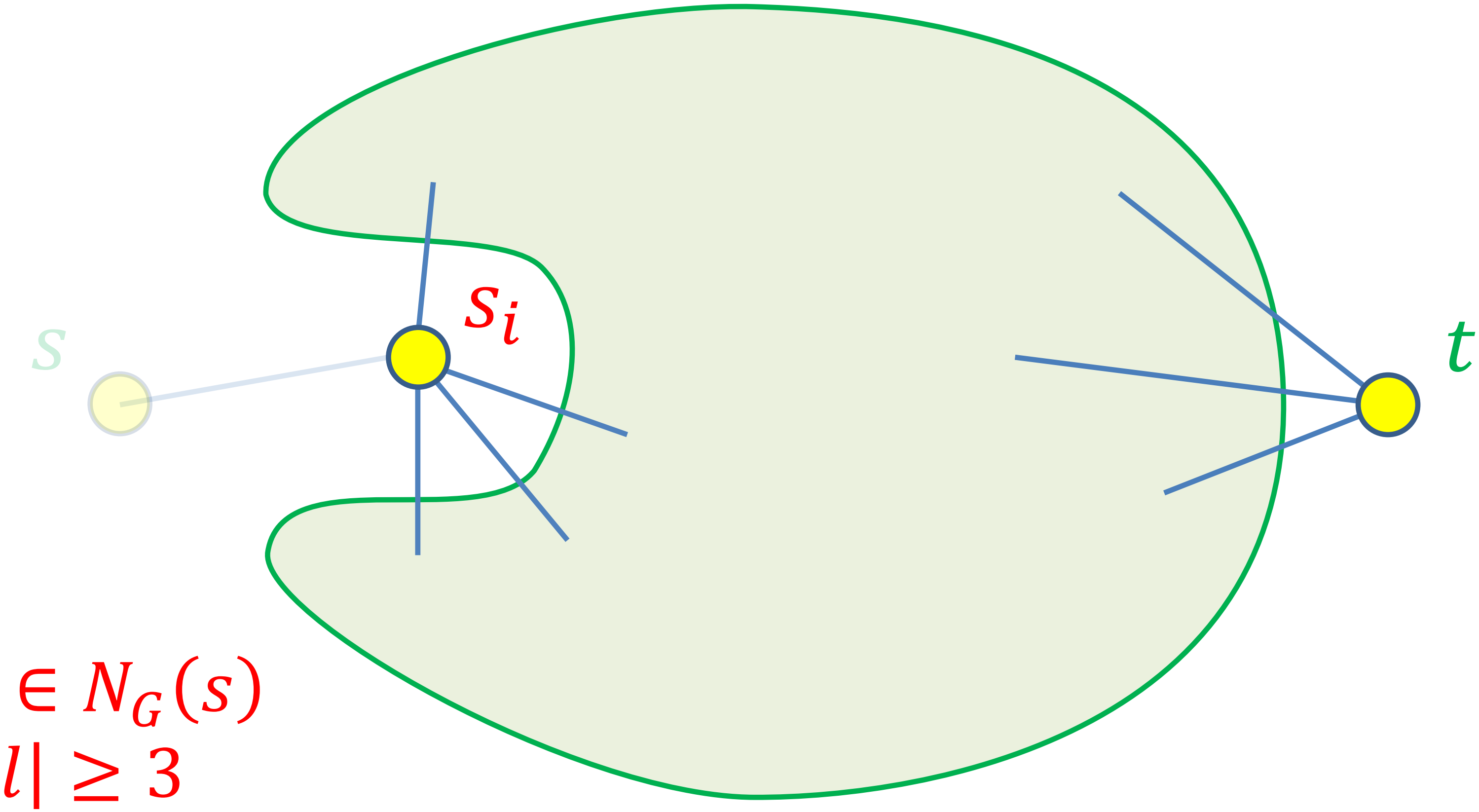


Finding s–t Paths of 3 Distinct Labels



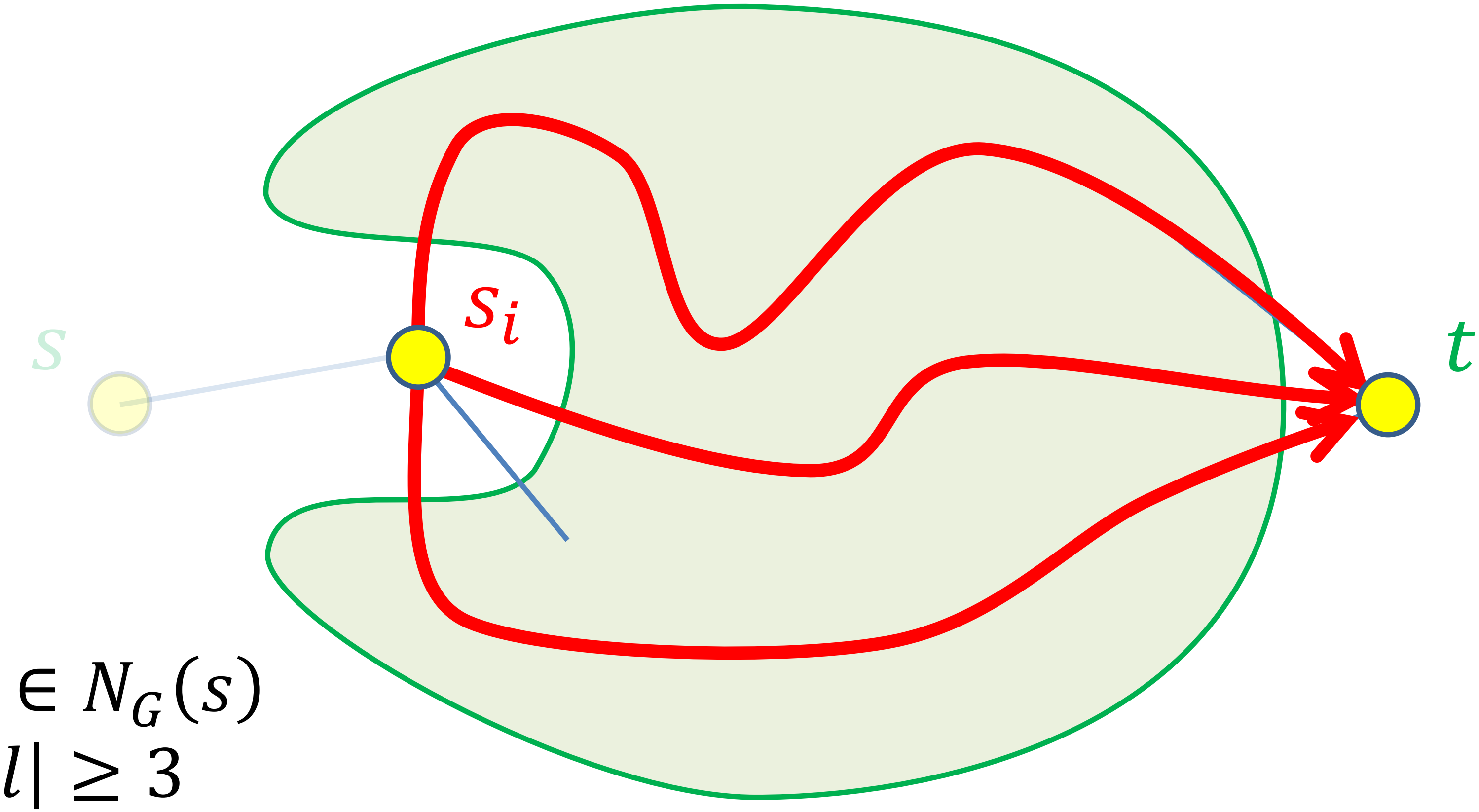
$|l| \leq 2$ or ≥ 3

Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$
$|l| \geq 3$

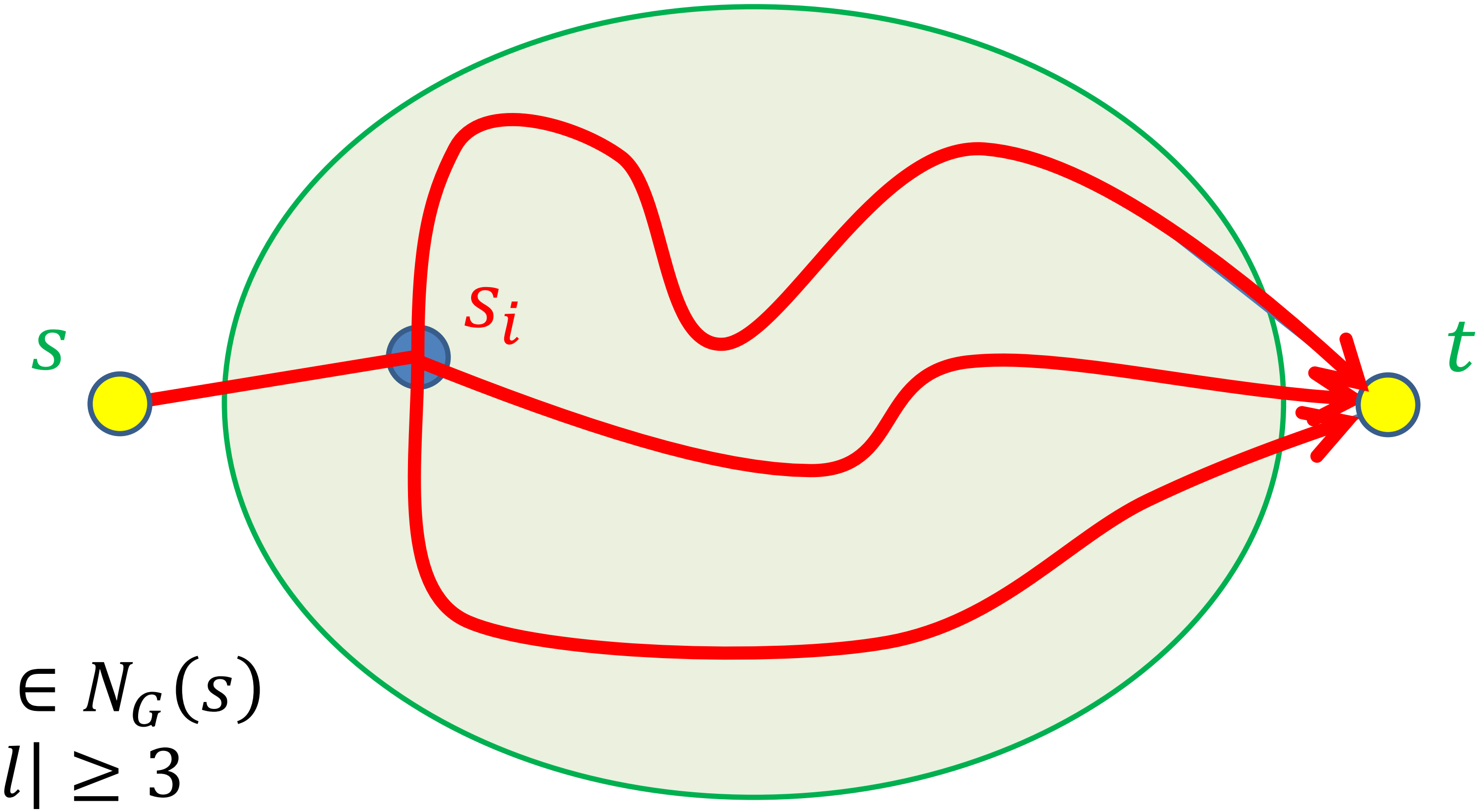
Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$
$|l| \geq 3$

Find 3 s_i–t paths **Recursively**

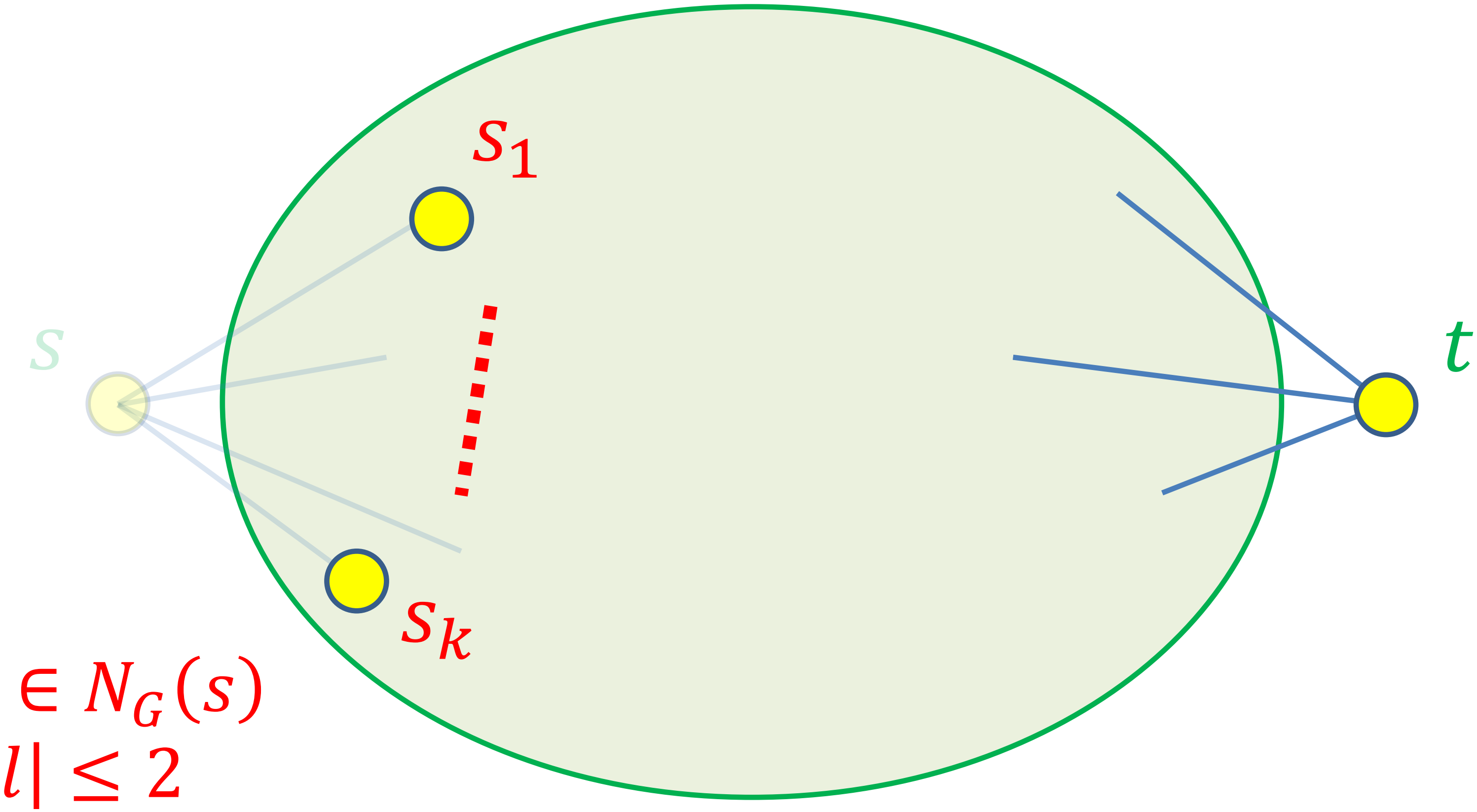
Finding s–t Paths of 3 Distinct Labels



$\exists s_i \in N_G(s)$
$|l| \geq 3$

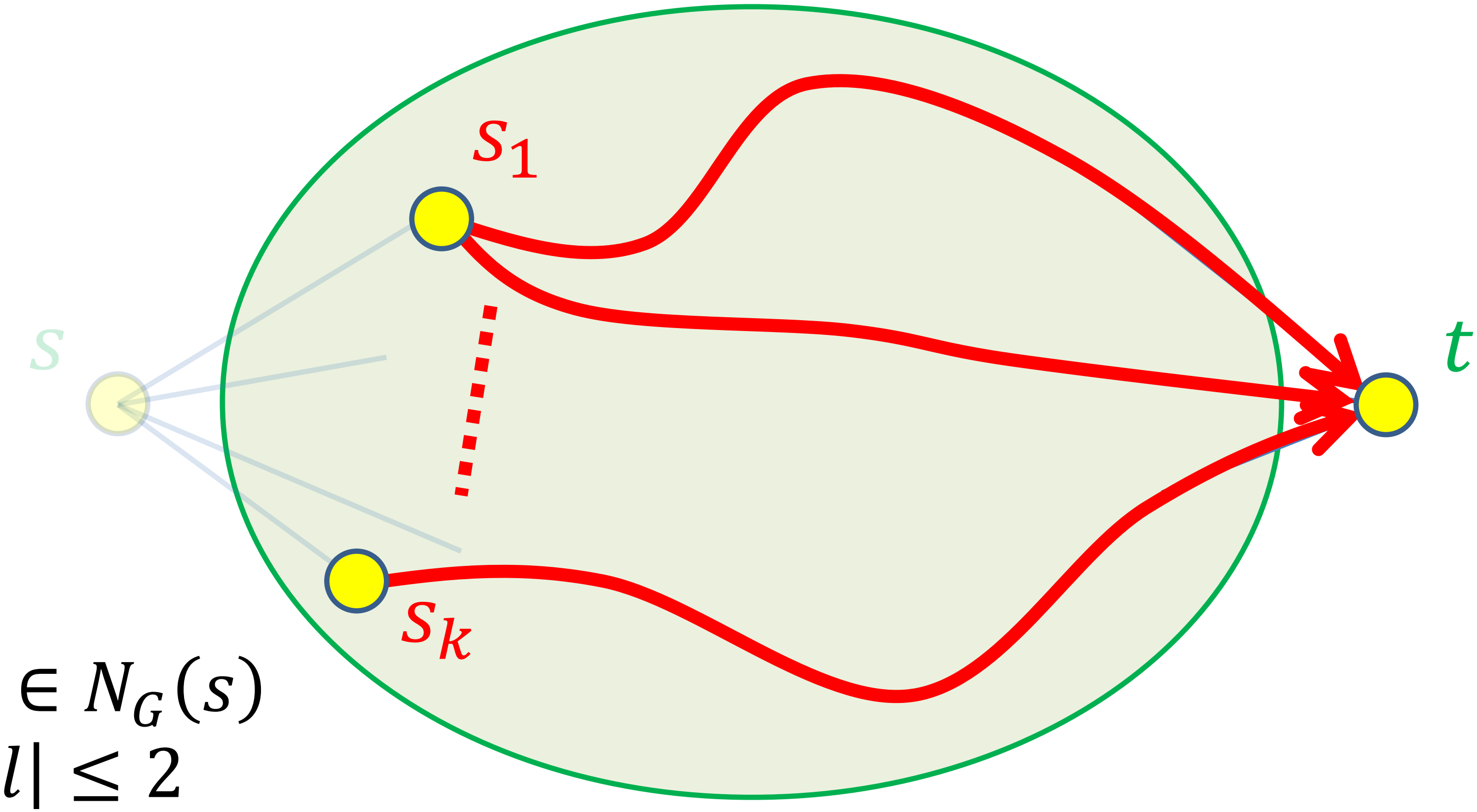
Find 3 s_i–t paths Recursively, and **Extend** them

Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

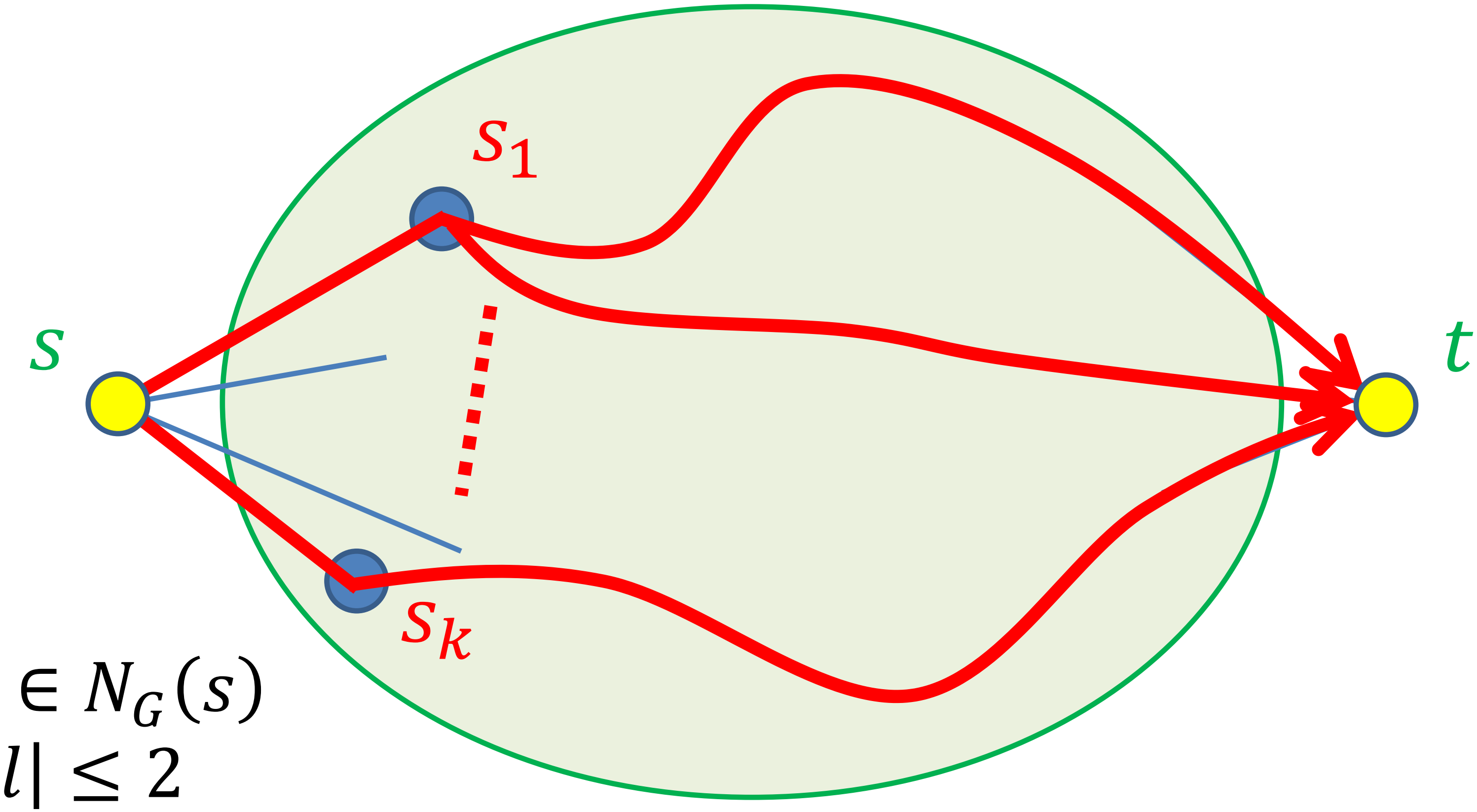
Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

Get s_i–t paths of **ALL** possible labels

Finding s–t Paths of 3 Distinct Labels



$\forall s_i \in N_G(s)$
$|l| \leq 2$

Get s_i–t paths of ALL possible labels, **Extend** and **Select**

Conclusion

- **Characterization** for a Group-Labeled Graph with **Exactly 2 Possible Labels** of s–t paths
  - **Polytime Testable**
  - Extends Char. for 2**-disjoint Paths** [Seymour 1980]
- **Algorithm** to find an s–t path **with 2 Labels Forbidden**
  - **Polytime**
  - **NOT** Depends on Group

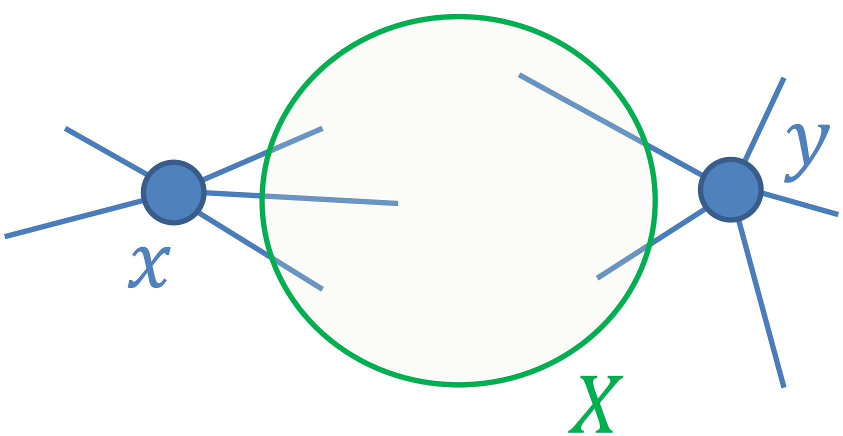
    Non-abelian or **Infinite** is OK
    If Group Operations in Const. time

2-contraction

2-contraction of $X \subseteq V \setminus \{s, t\}$ with $N_G(X) = \{x, y\}$

$$\overset{\text{def}}{\Updownarrow}$$

- Remove all vertices in X
- Add an edge from x to y
  with each label of an x–y path through X



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2-contraction of $X \subseteq V \setminus \{s, t\}$ with $N_G(X) = \{x, y\}$

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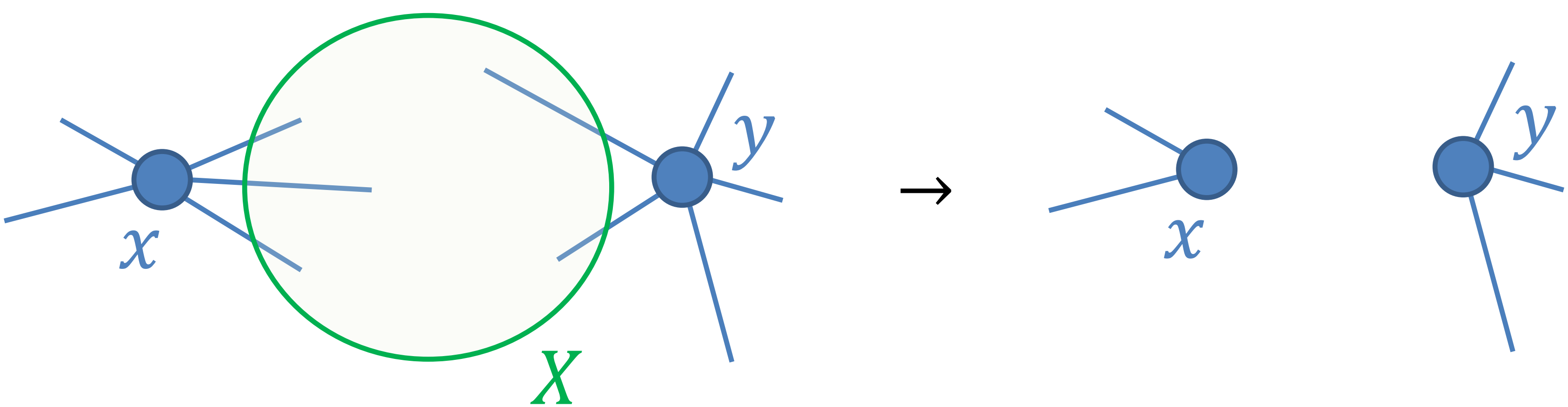

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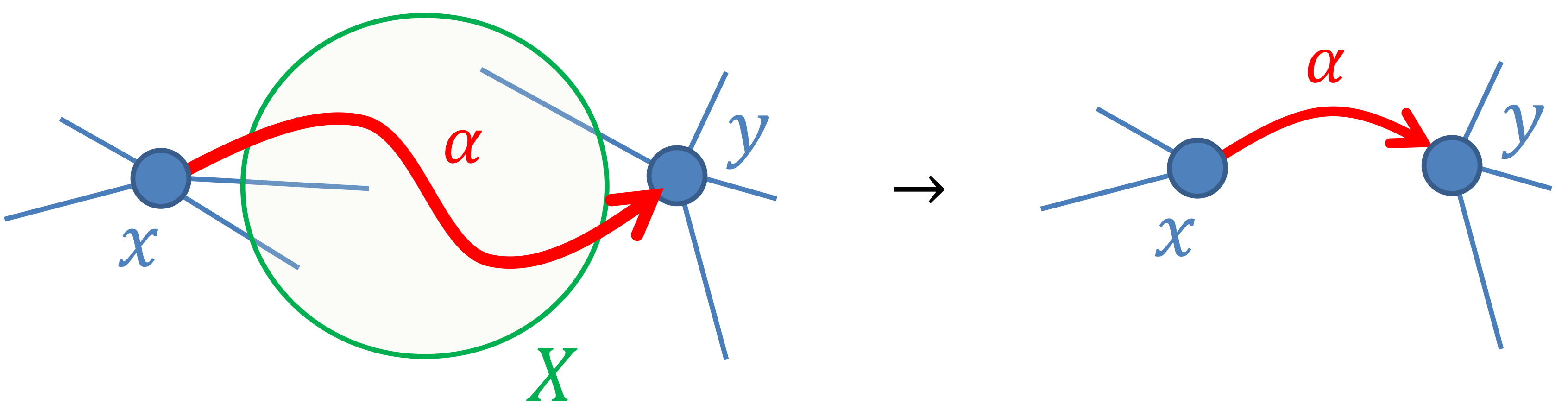

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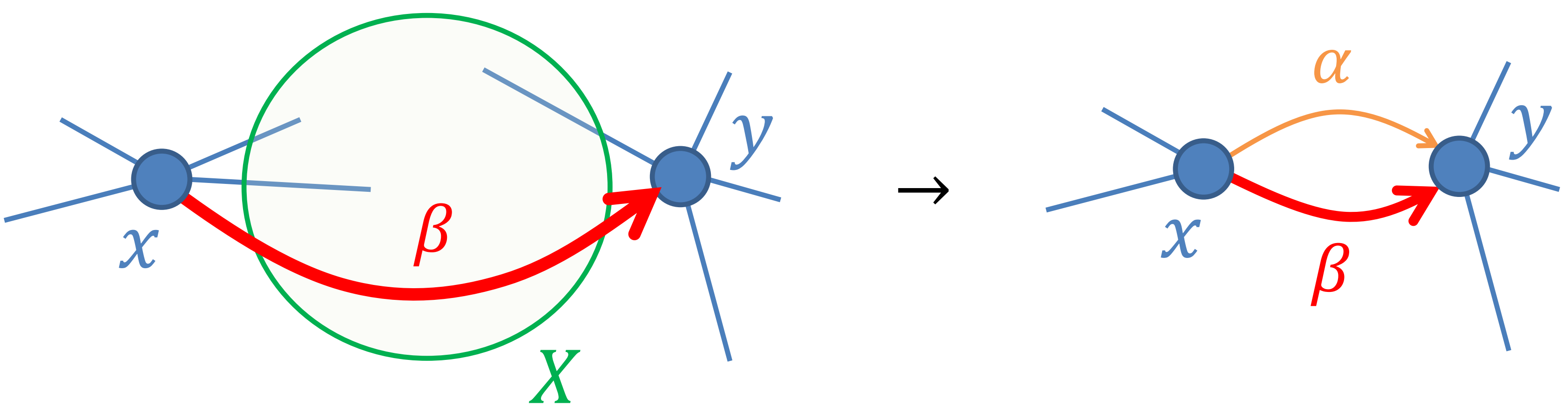
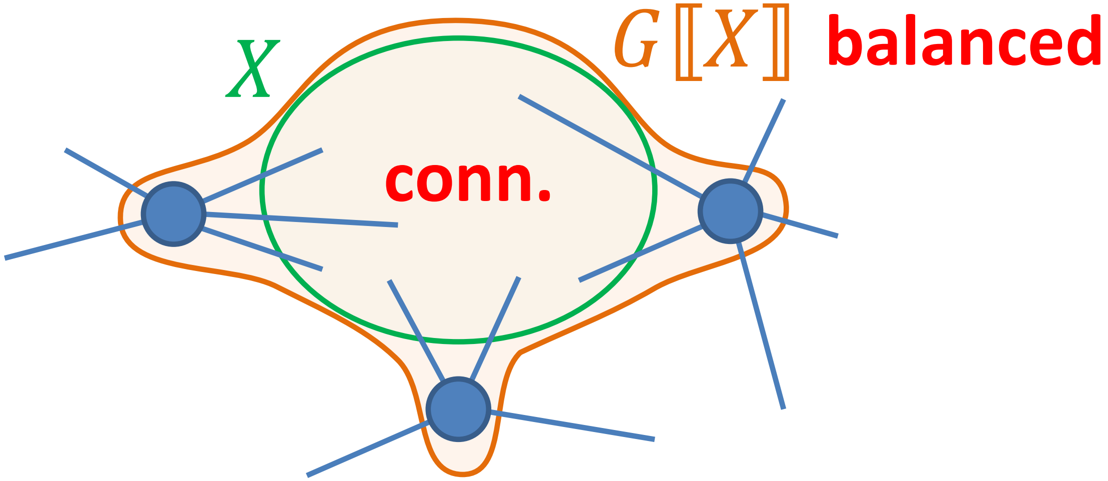


3-contraction

3-contraction of $X \subseteq V \setminus \{s, t\}$ with $|N_G(X)| = 3$
($G[X]$ is **connected** and $G[\![X]\!]$ is **balanced**)

def $\Updownarrow$

- Remove all vertices in X
- Add an edge xy w. label $l(G[\![X]\!]; x, y)$ $\left(\forall x, y \in N_G(X)\right)$

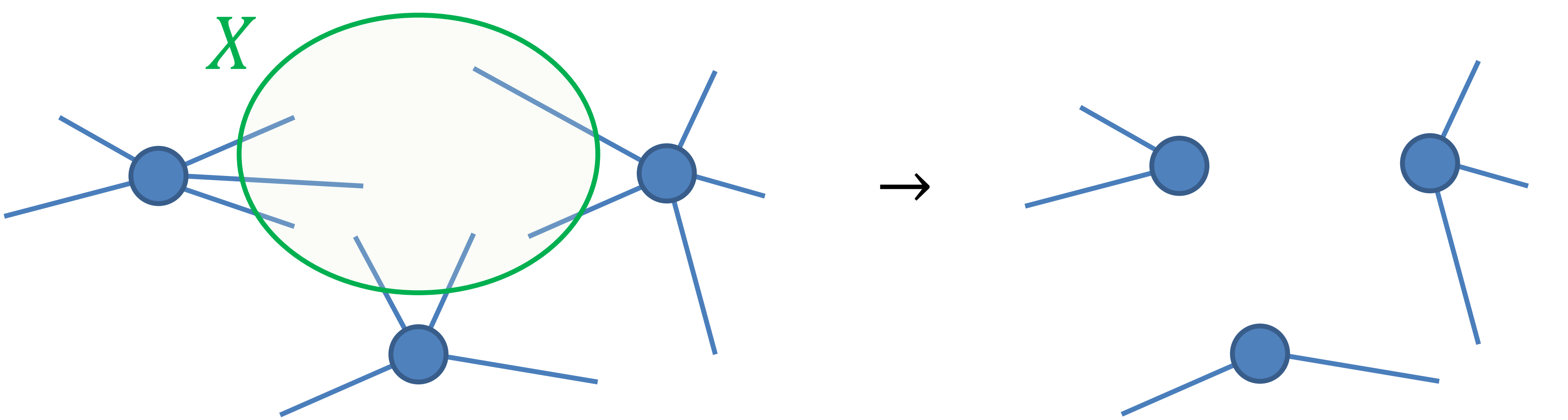


3-contraction

3-contraction of $X \subseteq V \setminus \{s, t\}$ with $|N_G(X)| = 3$
($G[X]$ is **connected** and $G[\![X]\!]$ is **balanced**)

def $\Updownarrow$

- **Remove all vertices in X**
- Add an edge xy w. label $l(G[\![X]\!]; x, y)$ $\big(\forall x, y \in N_G(X)\big)$



3-contraction

3-contraction of $X \subseteq V \setminus \{s,t\}$ with $|N_G(X)| = 3$
($G[X]$ is **connected** and $G[\![X]\!]$ is **balanced**)

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- Remove all vertices in X
- **Add an edge xy w. label $l(G[\![X]\!]; x, y)$** $\left(\forall x, y \in N_G(X)\right)$