Packing Non-zero A-paths via Matroid Matching

Shin-ichi Tanigawa[1], **Yutaro Yamaguchi**[2]

1. Kyoto University (RIMS), Japan.
2. University of Tokyo, Japan.

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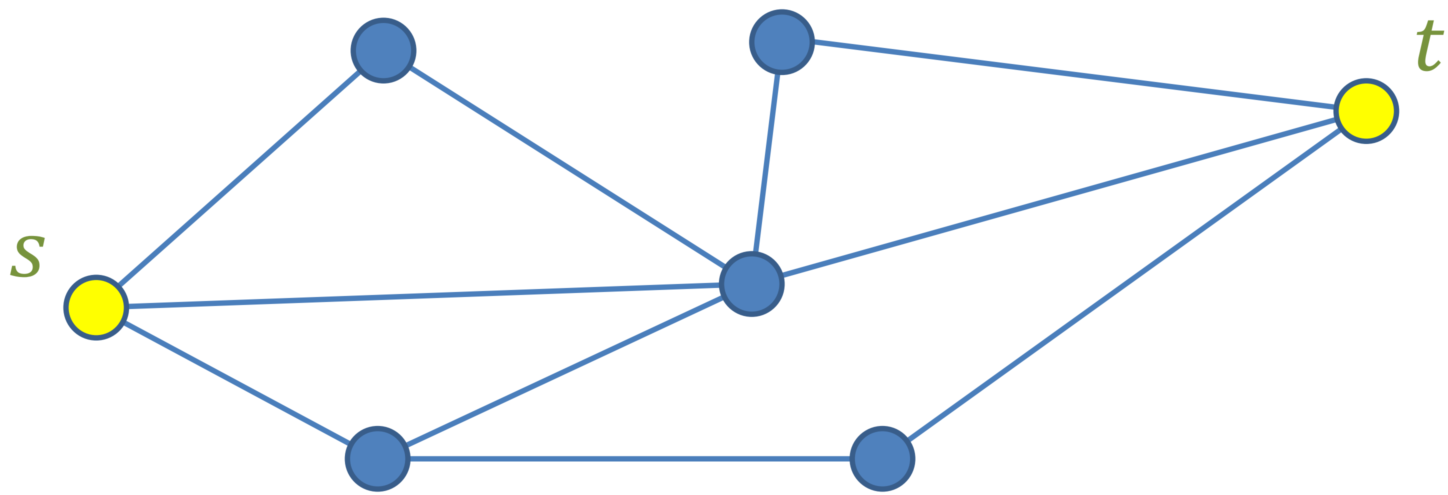
Menger's Disjoint Paths Problem

<u>Given</u> $G = (V, E)$: Undirected Graph

$s, t \in V$: Distinct Terminals



<u>Find</u> **Maximum Number of Disjoint s–t paths**

Menger’s Disjoint Paths Problem

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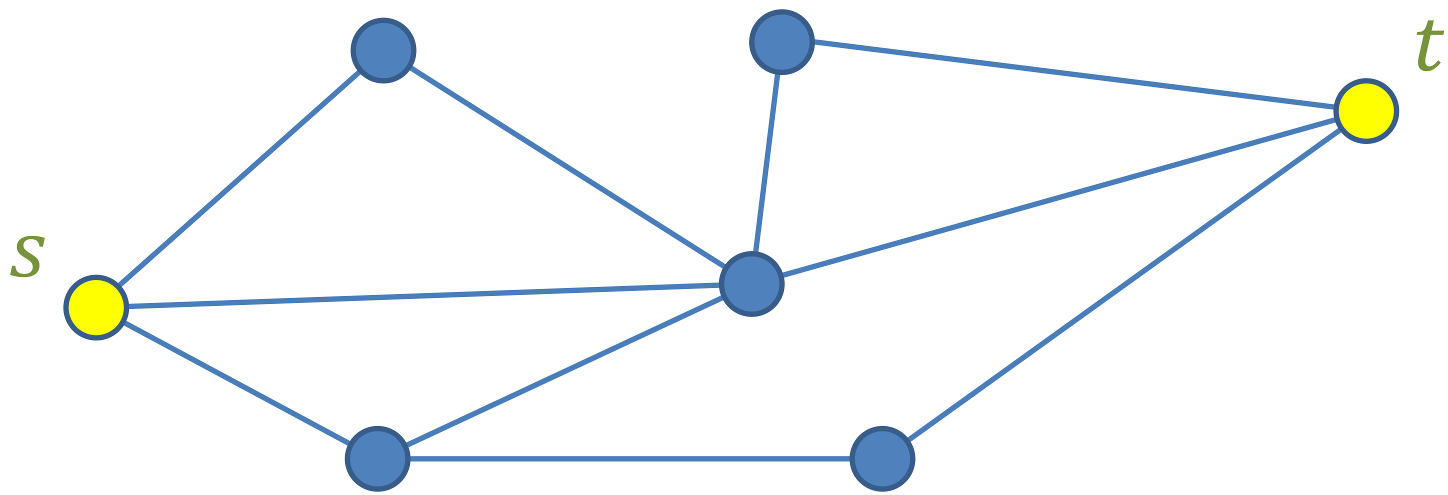

Find **Maximum Number of Disjoint** s–t **paths**

Edge or **Vertex**

Menger's Disjoint Paths Problem

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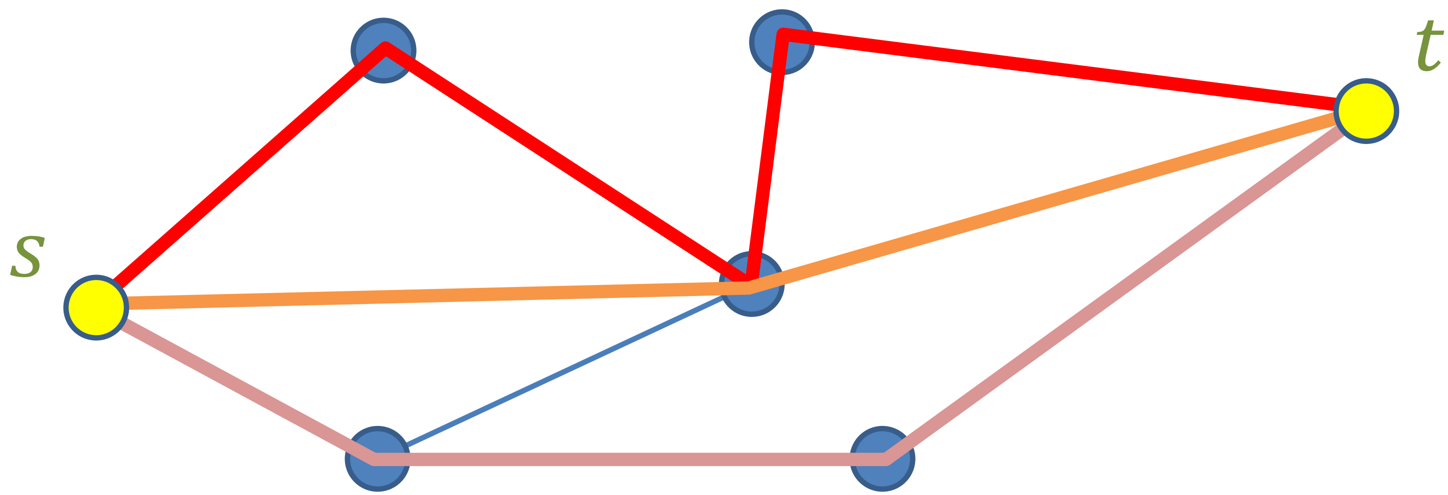

Find **Maximum Number of Disjoint** s–t **paths**

Don't Share **Edge**

Menger's Disjoint Paths Problem

Given $G = (V, E)$: Undirected Graph

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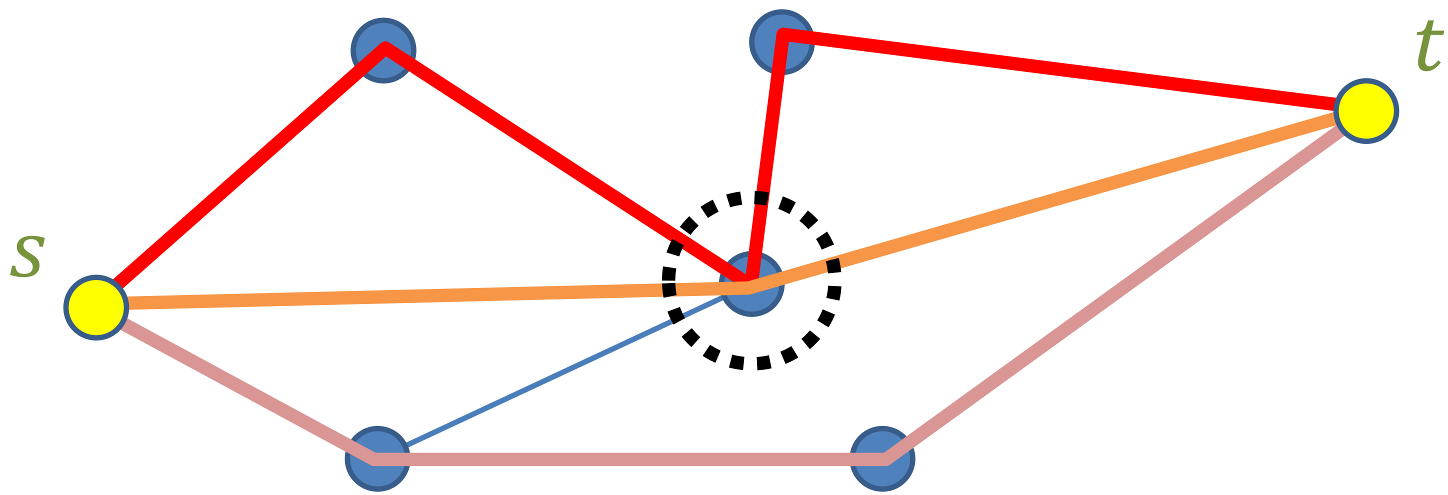

Find **Maximum Number of Disjoint s–t paths**

Don't Share **Vertex**

Menger's Disjoint Paths Problem

<u>Given</u> $G = (V, E)$: Undirected Graph

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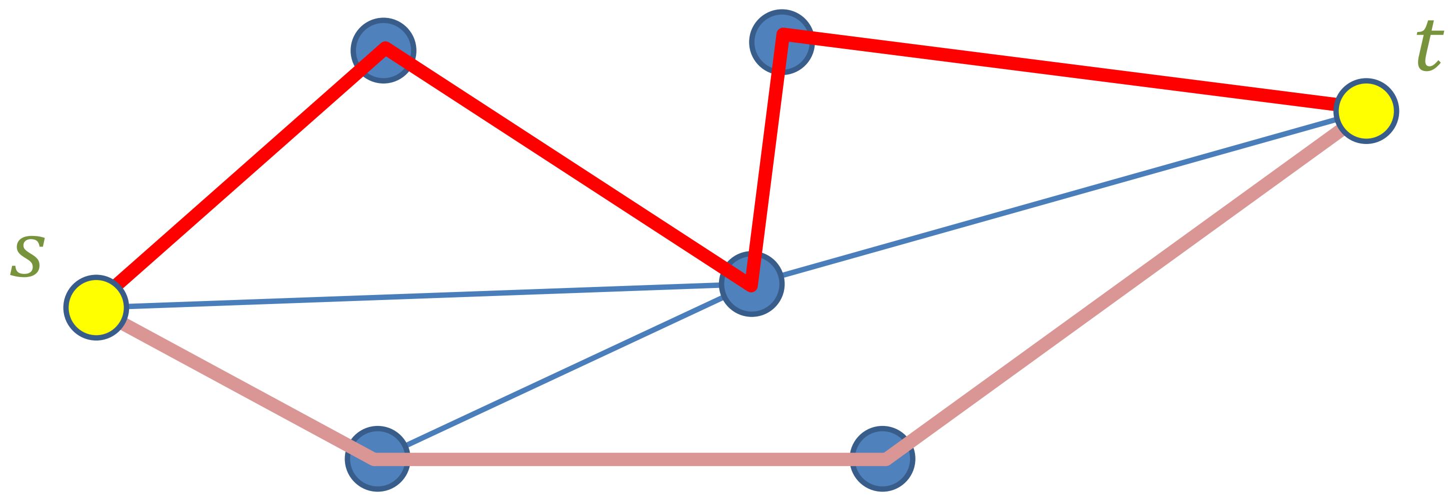
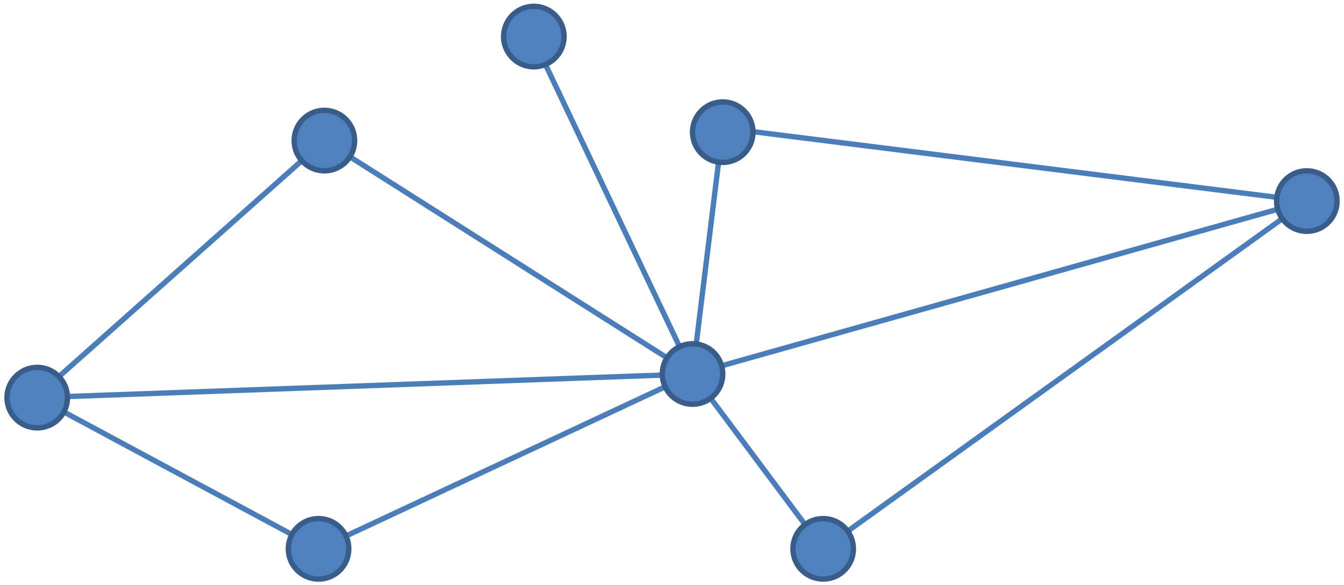


<u>Find</u> **Maximum Number of Disjoint s–t paths**

Don't Share **<u>Vertex</u>**

(Non-bipartite) Maximum Matching

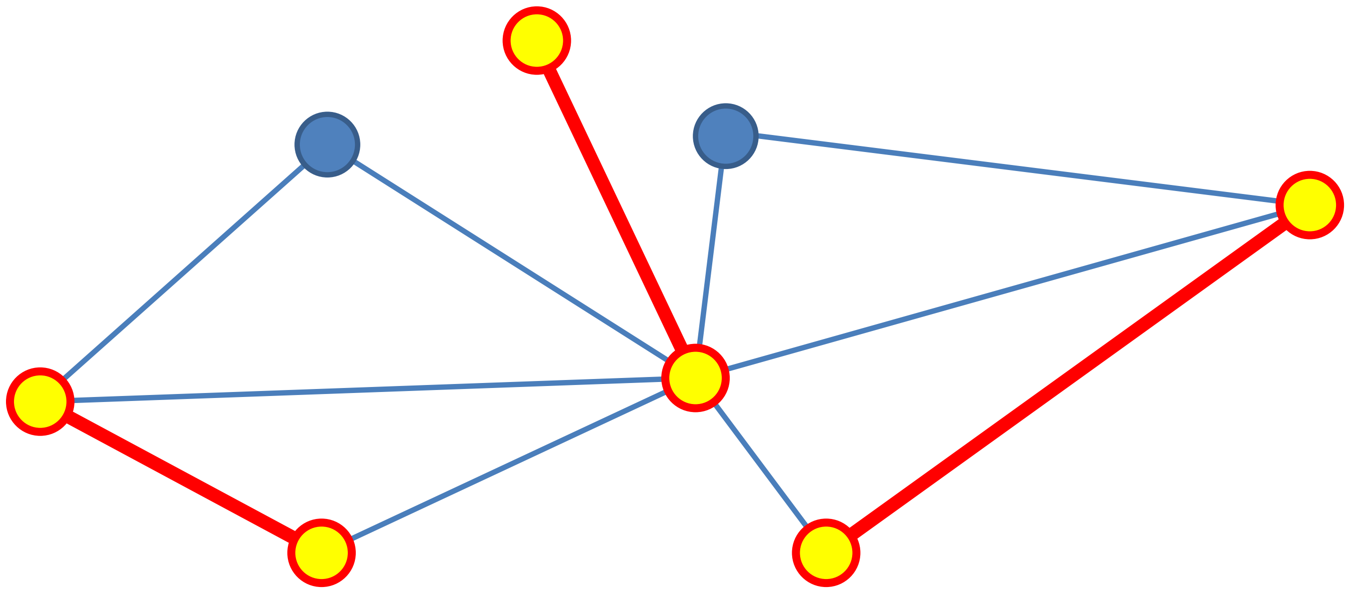
Given $G = (V, E)$: Undirected Graph



Find **Maximum Matching**

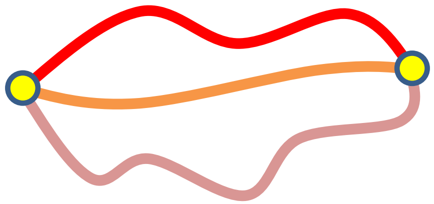
(Non-bipartite) Maximum Matching

Given $G = (V, E)$: Undirected Graph

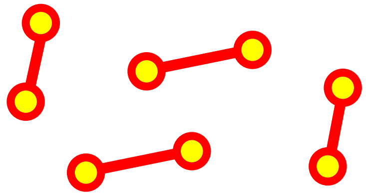


Find **Maximum** **Matching** **Edges**
Not Sharing End Vertices

Overview

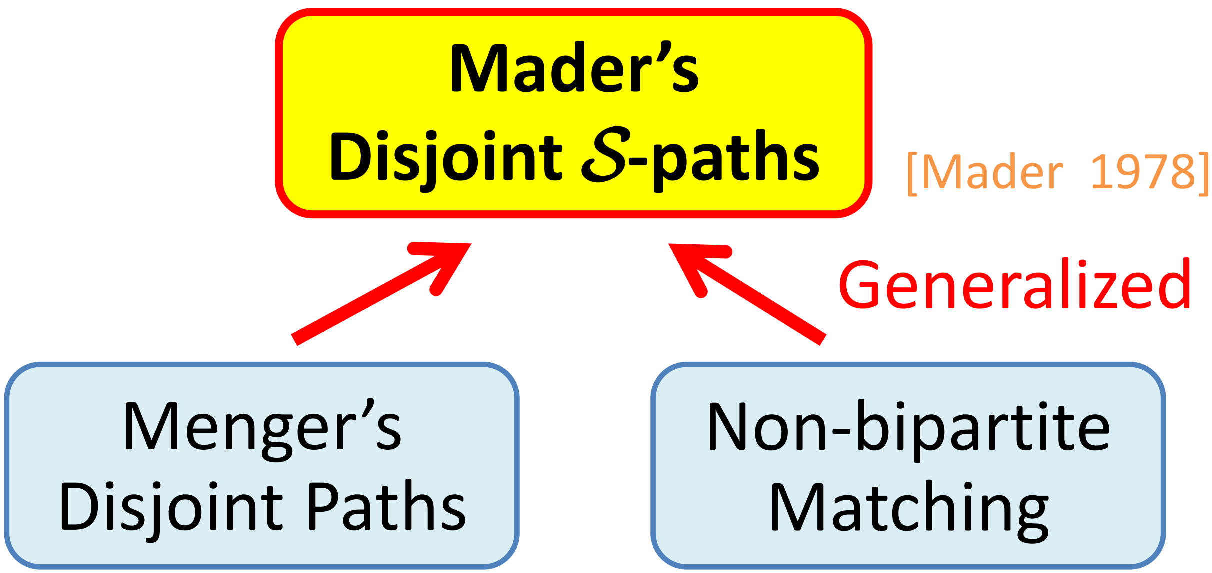


Menger's
Disjoint Paths

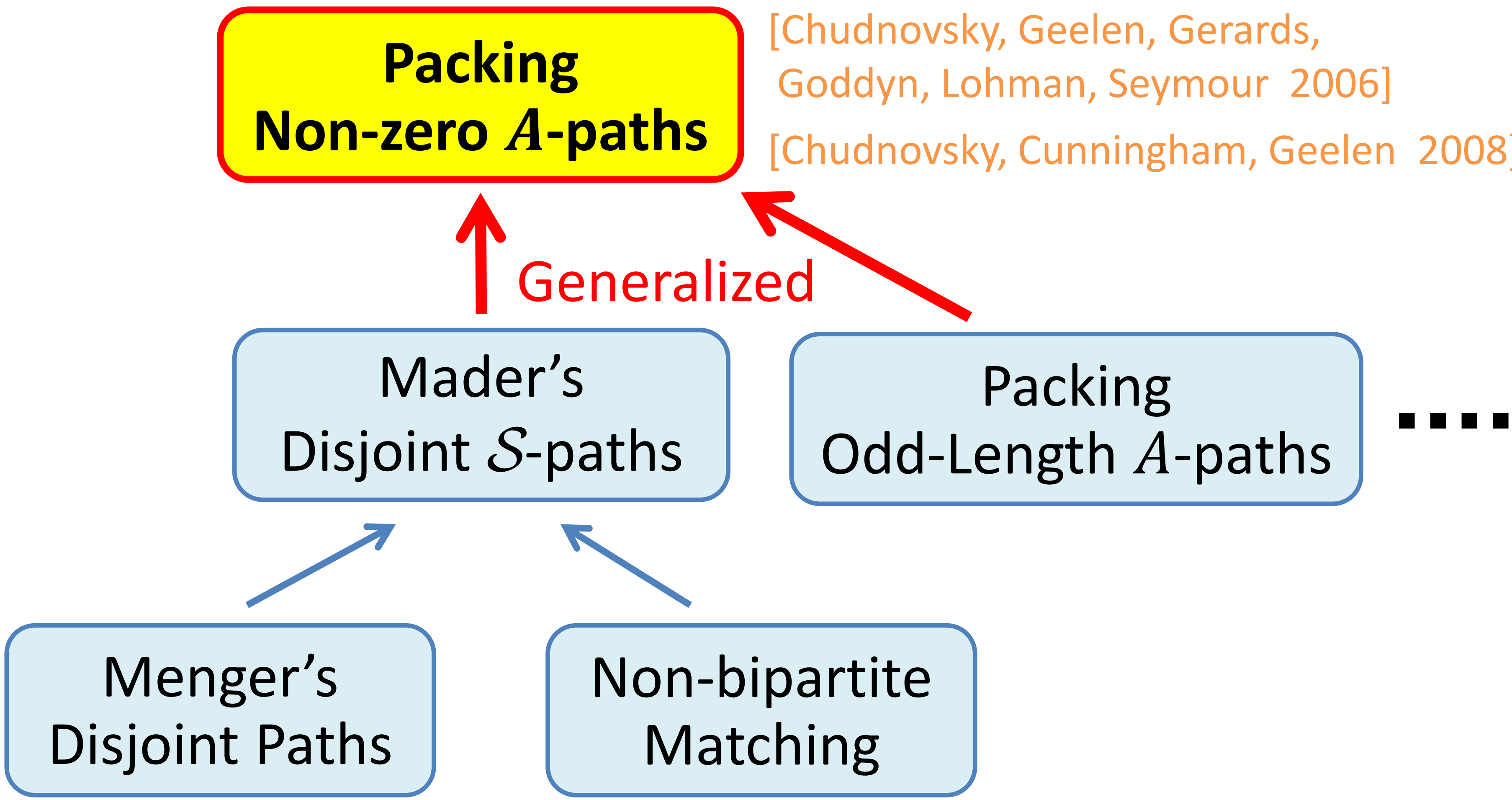


Non-bipartite
Matching

Overview


Mader's
Disjoint $\mathcal{S}$-paths
[Mader 1978]
Generalized
Menger's
Disjoint Paths
Non-bipartite
Matching

Overview


Packing
Non-zero A-paths
[Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]
[Chudnovsky, Cunningham, Geelen 2008]
Generalized
Mader's
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Odd-Length A-paths
....
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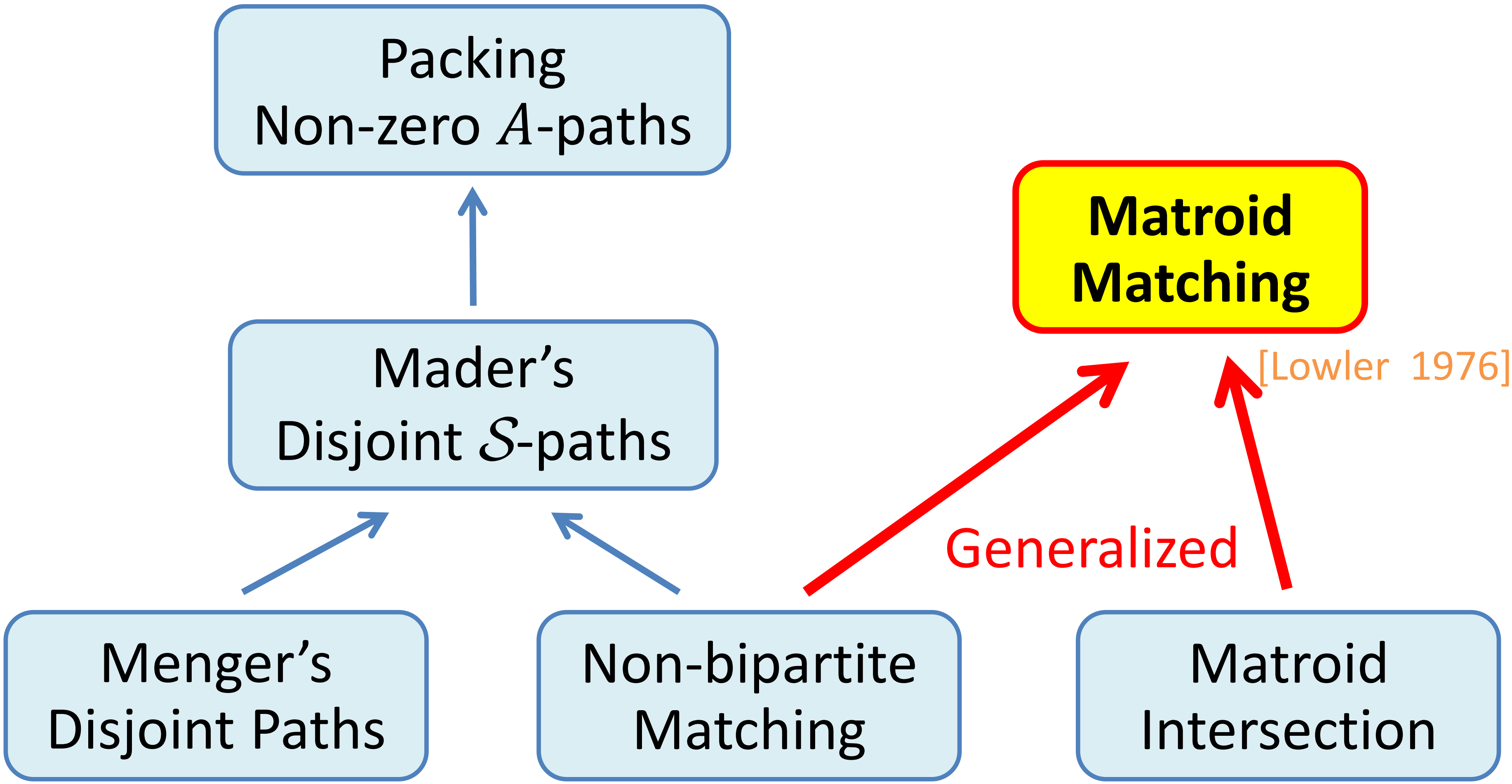
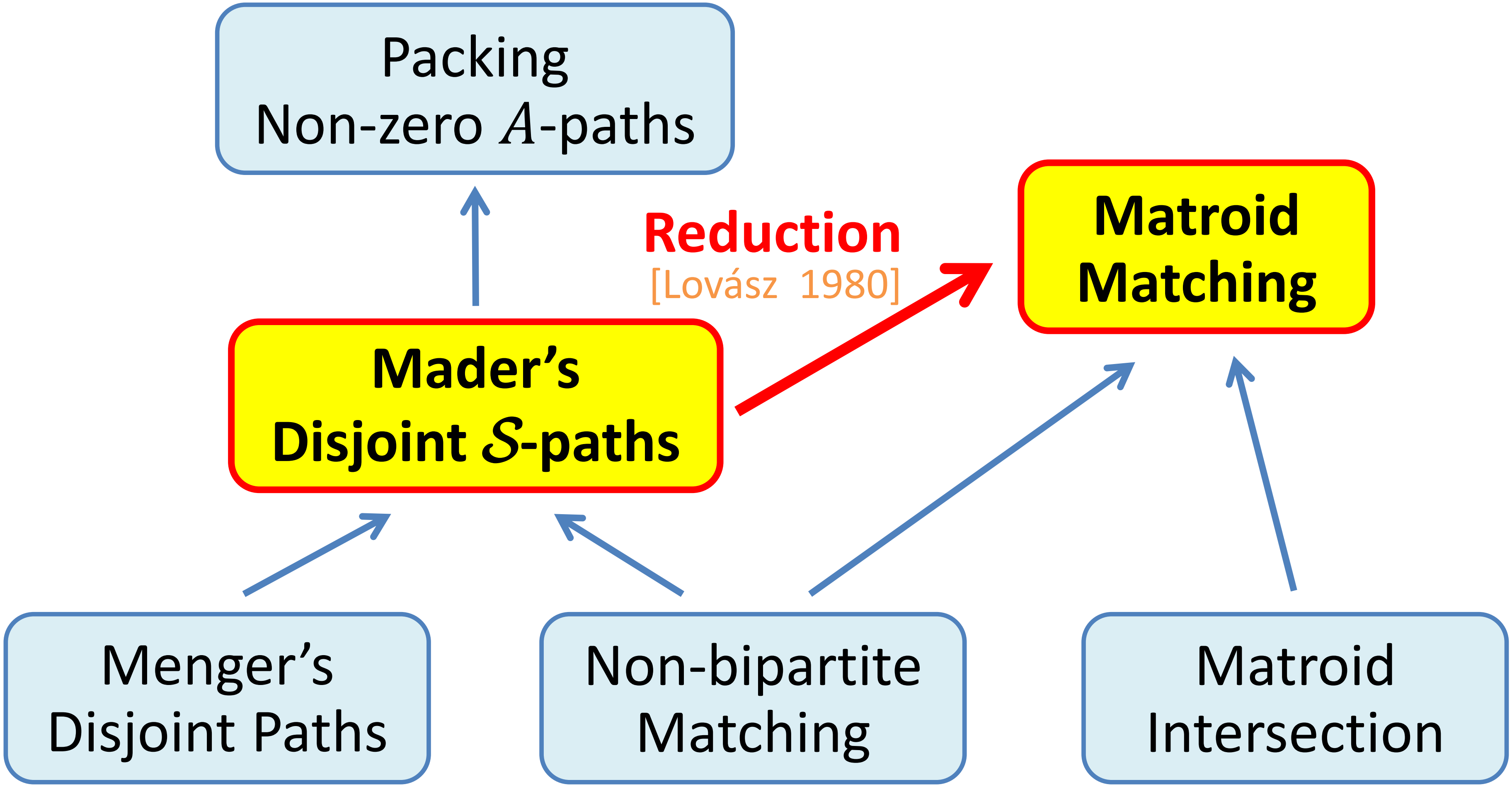

Packing
Non-zero A-paths
Matroid
Matching
Mader's
Disjoint $\mathcal{S}$-paths
[Lowler 1976]
Generalized
Menger's
Disjoint Paths
Non-bipartite
Matching
Matroid
Intersection

Overview



Packing
Non-zero A-paths
Reduction
[Lovász 1980]
Matroid
Matching
Mader's
Disjoint $\mathcal{S}$-paths
Menger's
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Non-bipartite
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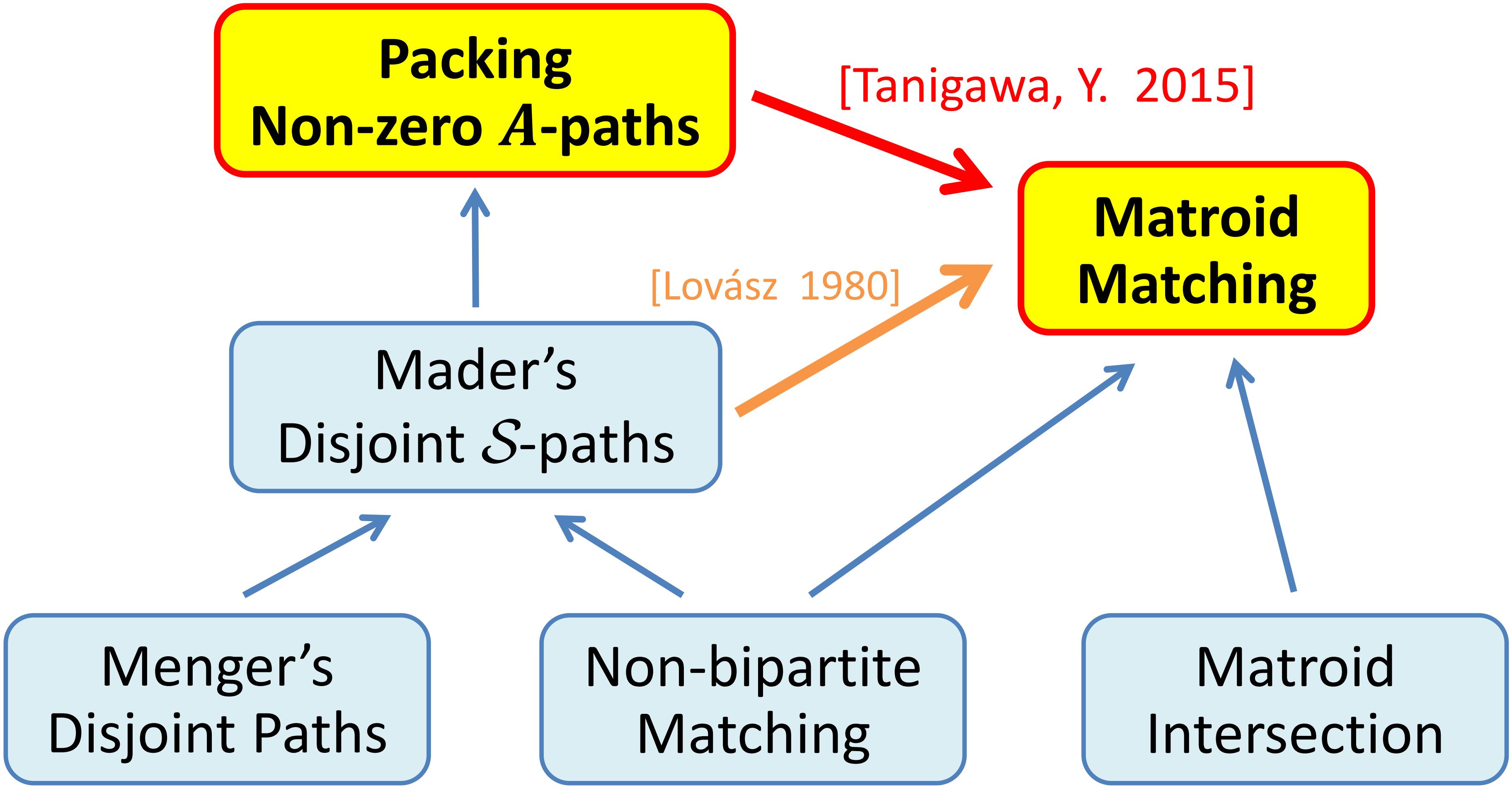
Matroid
Intersection

Overview



Packing
Non-zero A-paths
[Tanigawa, Y. 2015]
Matroid
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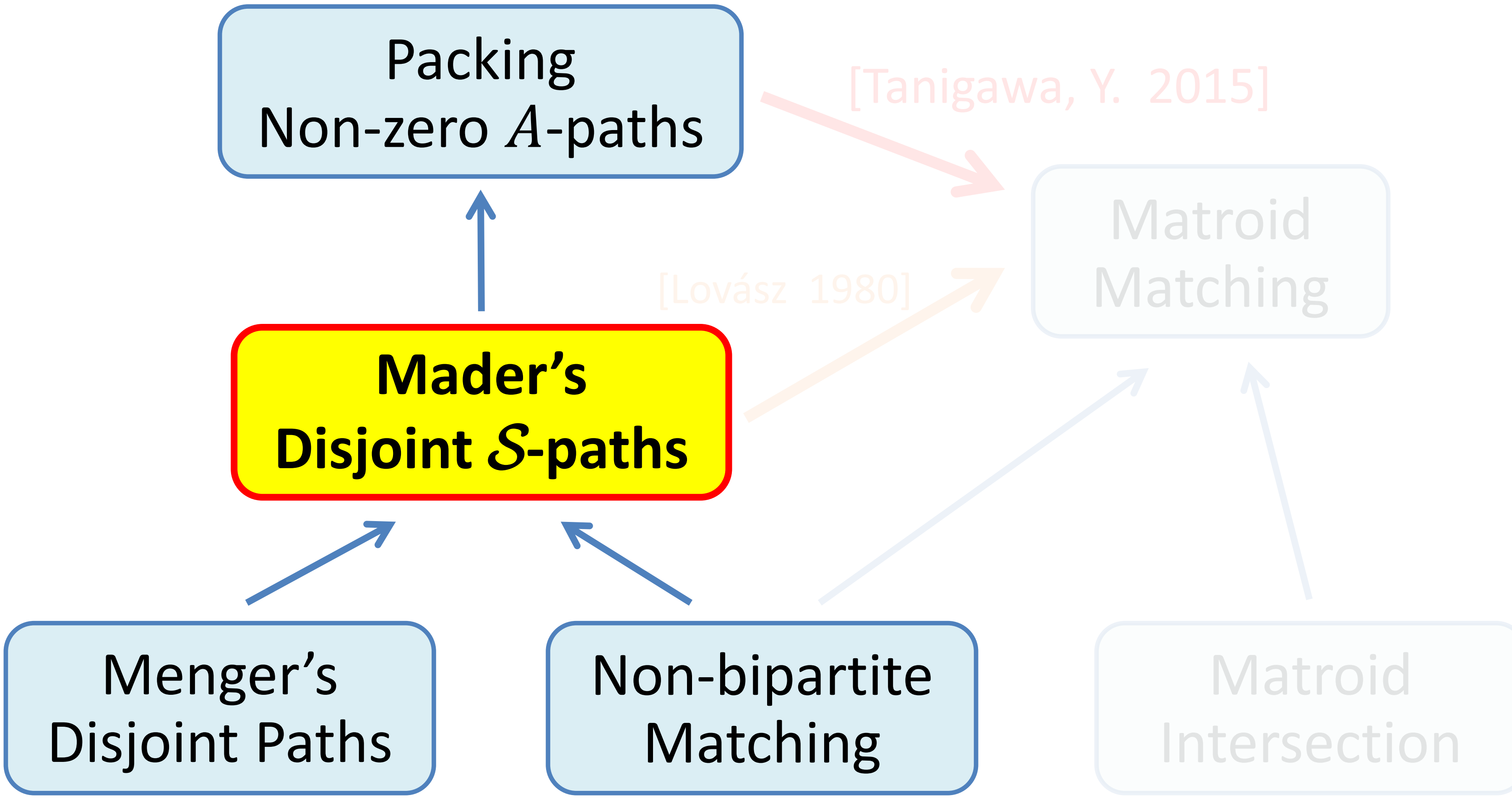

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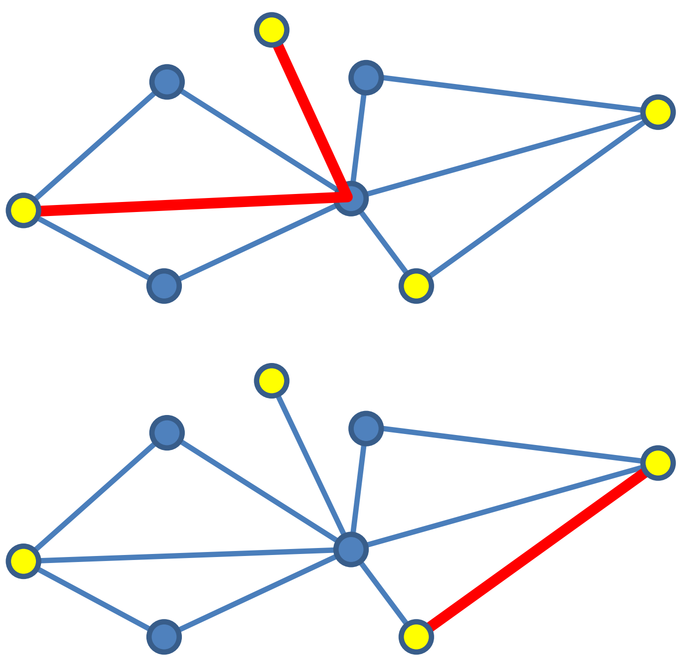
Intersection

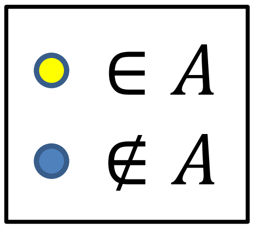
A-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

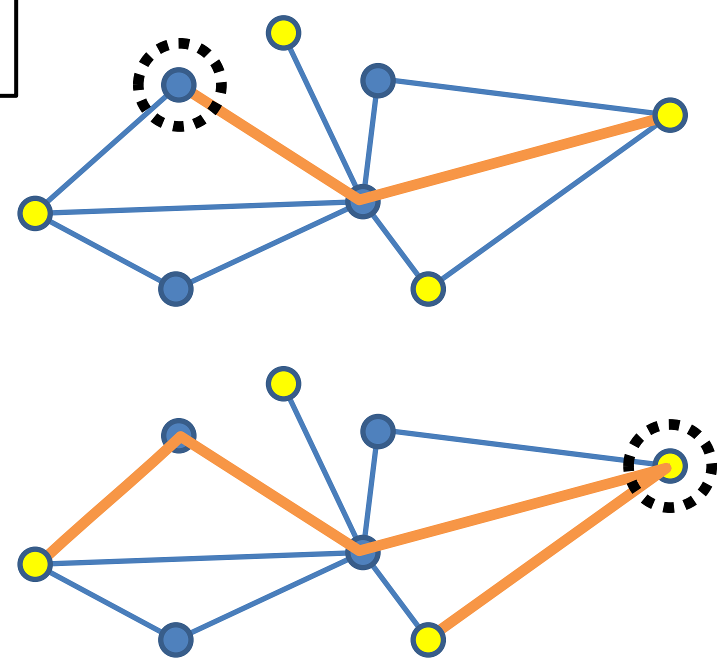
$A \subseteq V$: Terminal Set

A-paths





NOT A-paths



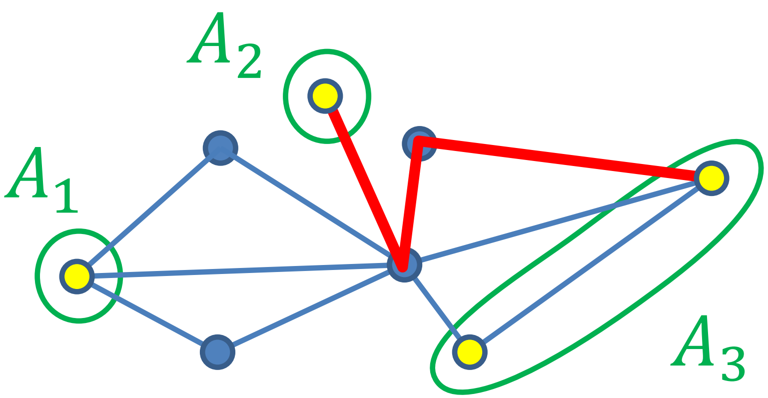
A-paths and $\mathcal{S}$-paths

$G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set

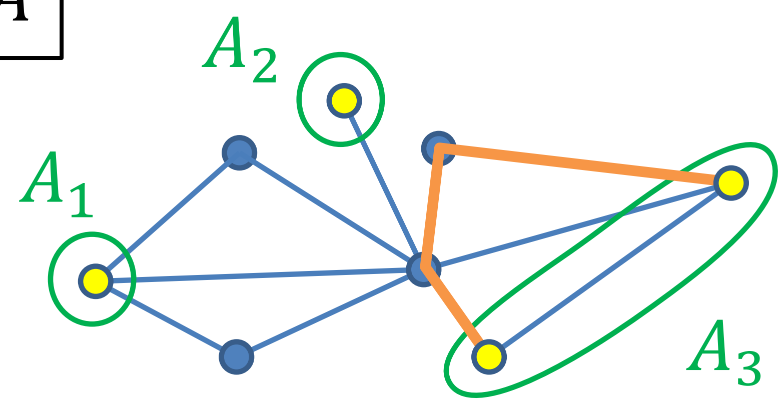
$\mathcal{S} = \{A_1, A_2, \ldots, A_k\}$: Partition of A

$\mathcal{S}$-path



- (yellow) $\in A$
- (blue) $\notin A$

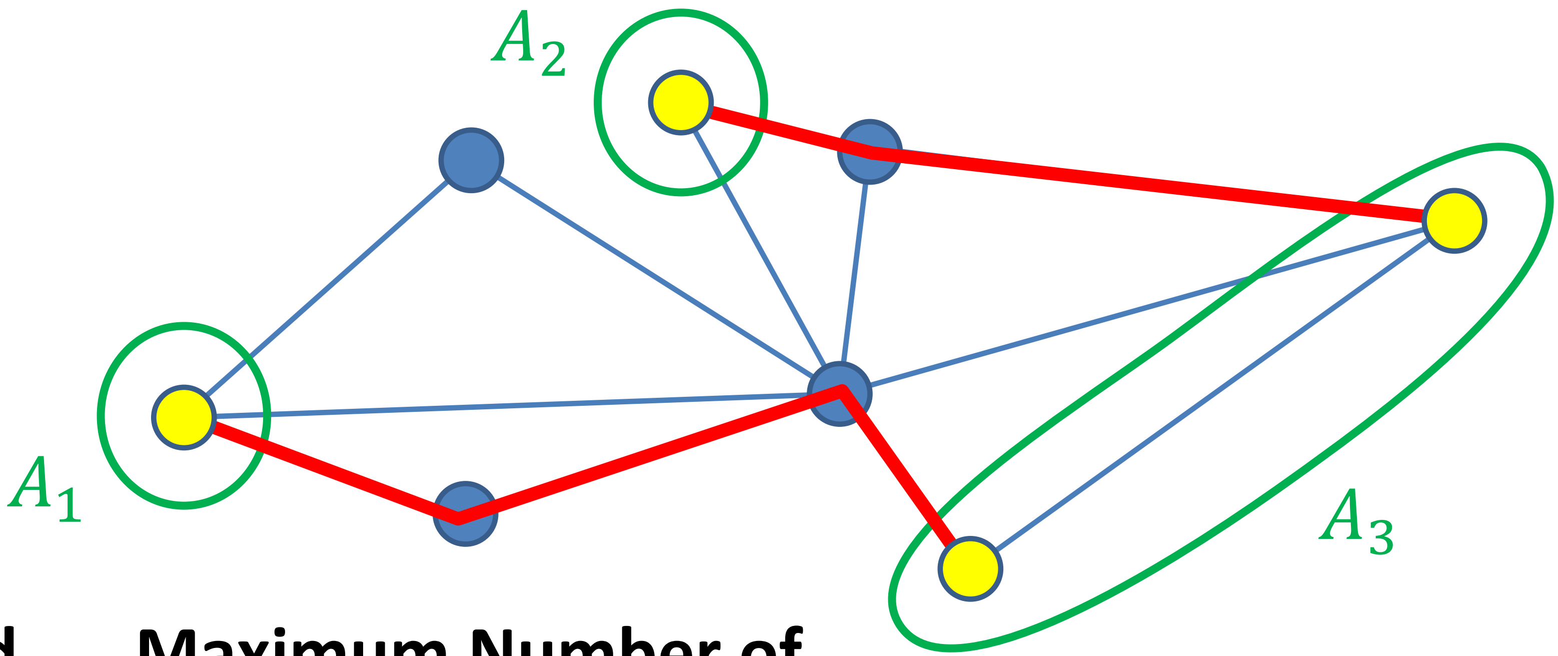
NOT $\mathcal{S}$-path



Mader's Disjoint $\mathcal{S}$-paths Problem

Given $G = (V, E)$: Undirected Graph

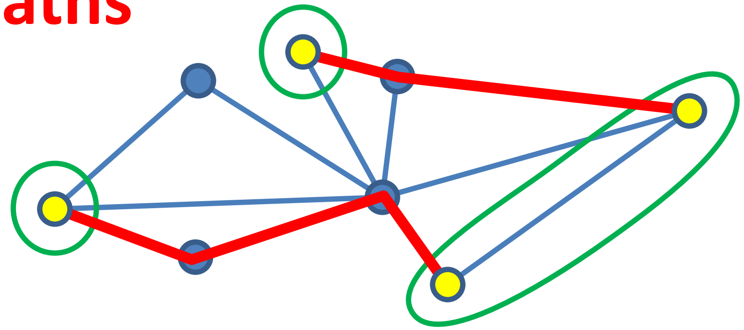
$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of A



Find **Maximum Number of**
Fully Vertex-Disjoint $\mathcal{S}$-paths

Mader's Disjoint $\mathcal{S}$-paths Problem

<u>Find</u> **Maximum Number of**
<u>Fully Vertex-Disjoint</u> $\mathcal{S}$-paths



- Min-Max Formula [Mader 1978]

Mader's Disjoint $\mathcal{S}$-paths Problem

<u>Find</u> **Maximum Number of**
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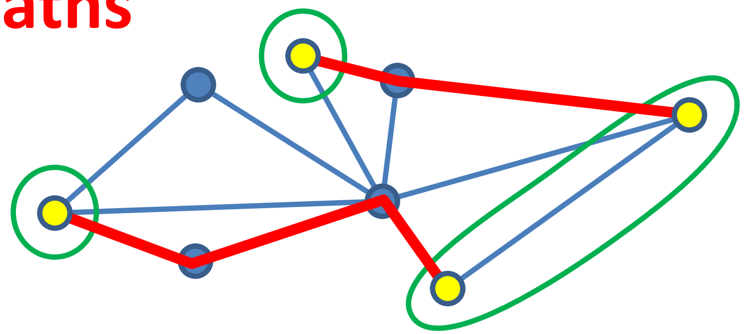
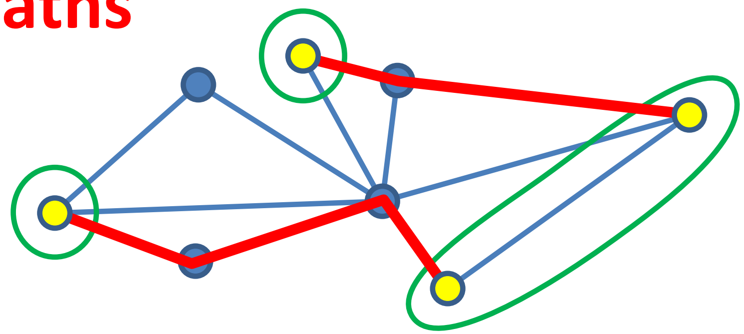


- Min-Max Formula [Mader 1978]
- Reduction to **<u>Matroid Matching</u>** [Lovász 1980]
  - Alternative Proof for Mader's Theorem
  - **<u>Polytime Solvability</u>** [Lovász 1981]

Mader's Disjoint $\mathcal{S}$-paths Problem

Find **Maximum Number of**
Fully Vertex-Disjoint **$\mathcal{S}$-paths**



- Min-Max Formula [Mader 1978]
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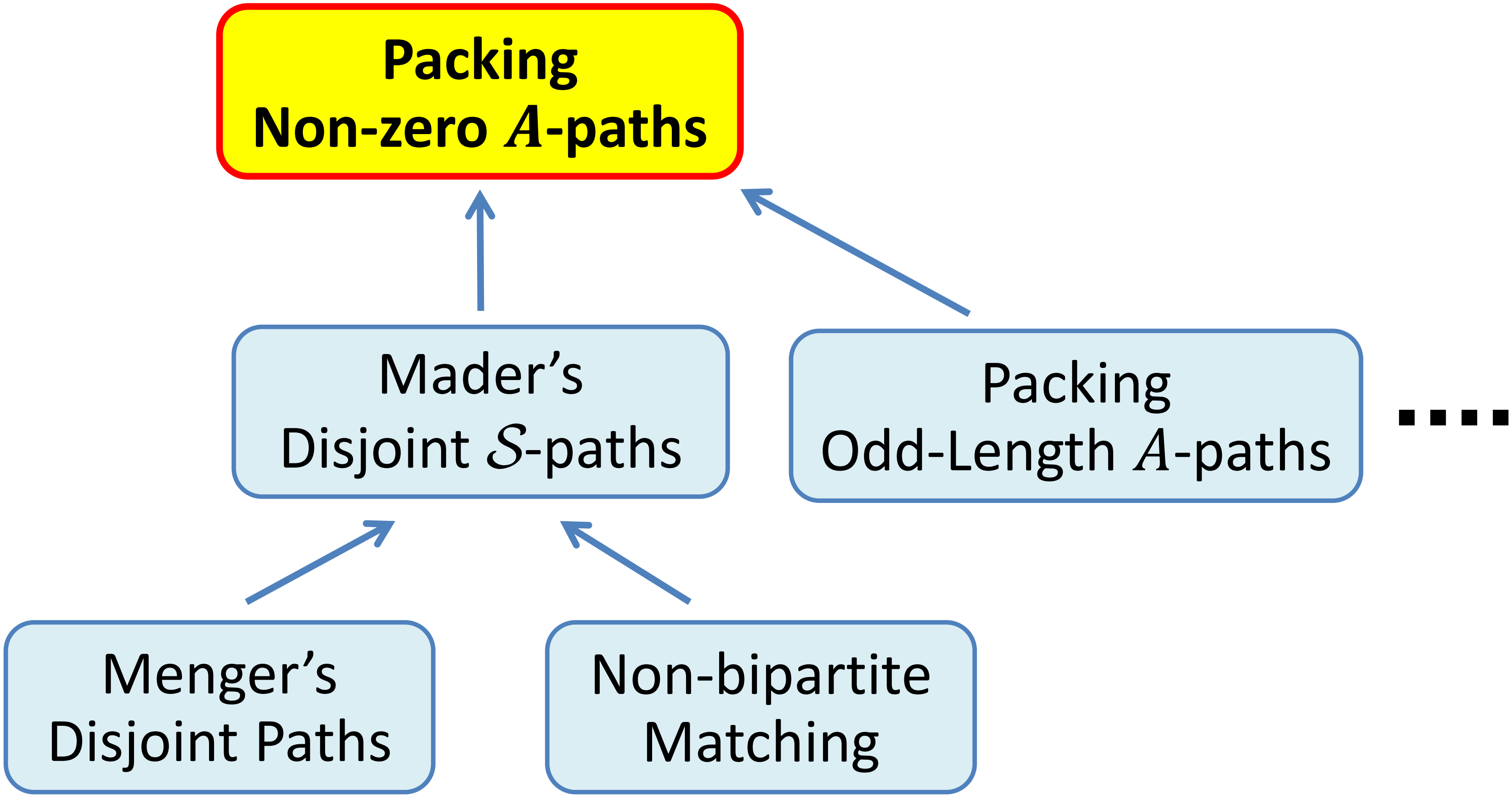
  - Alternative Proof for Mader's Theorem
  - **Polytime Solvability** [Lovász 1981]

  → Improved via **Linear Representation** [Schrijver 2003]

Overview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Matching

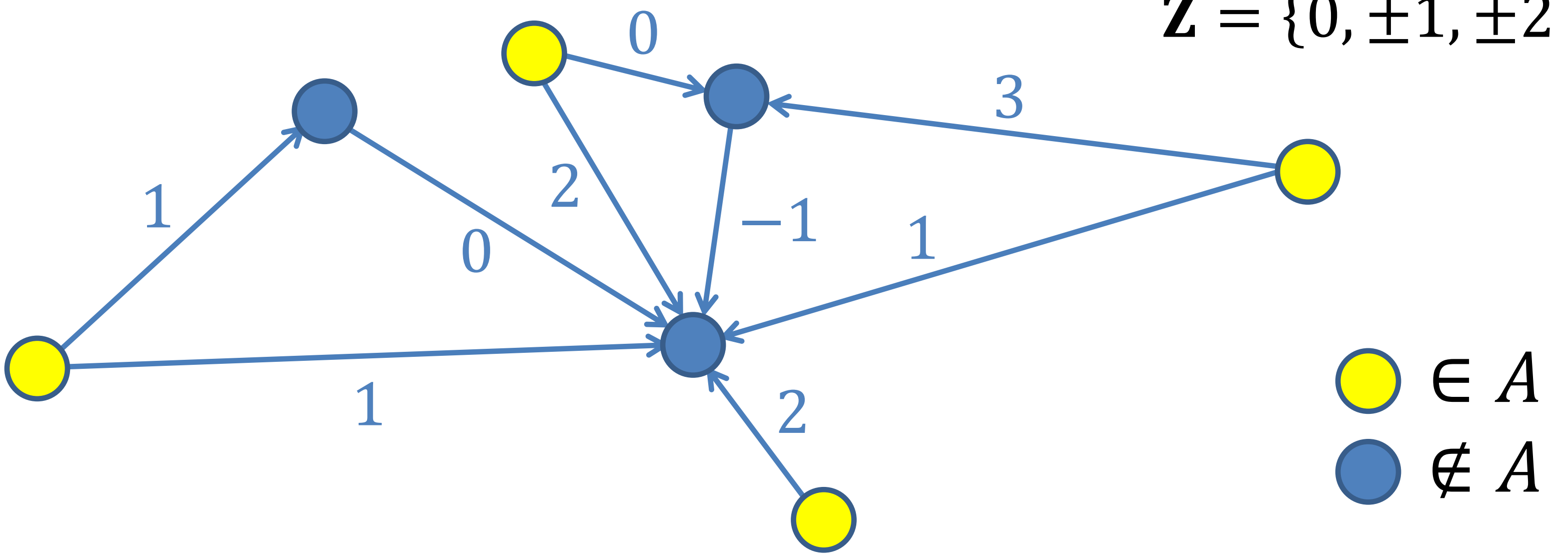
Packing Non-zero A-paths

Given $G = (V, E)$: Group-Labeled Graph

$A \subseteq V$: Terminal Set

$\mathbf{Z}$-Labeled Graph

$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$



Find **Maximum Number of**

Fully Vertex-Disjoint Non-zero A**-paths**

Packing Non-zero A-paths

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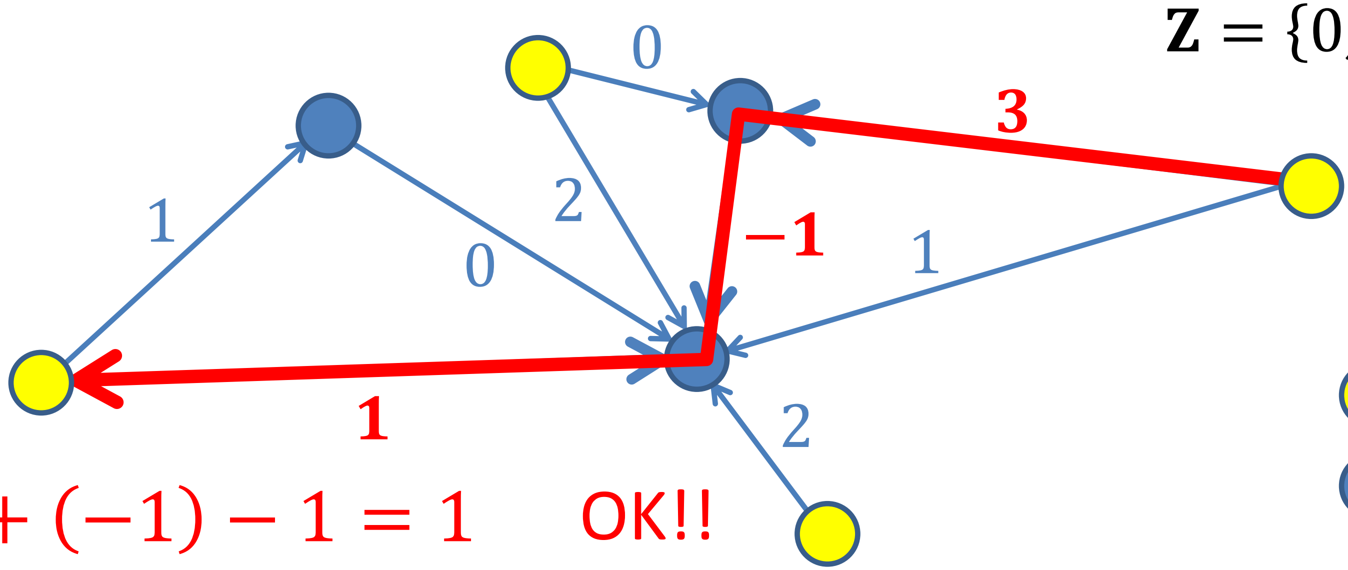

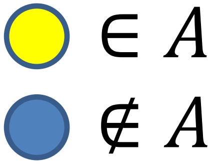

$3 + (-1) - 1 = 1$ OK!!

Find **Maximum Number of**

Fully Vertex-Disjoint Non-zero A-paths

Packing Non-zero A-paths

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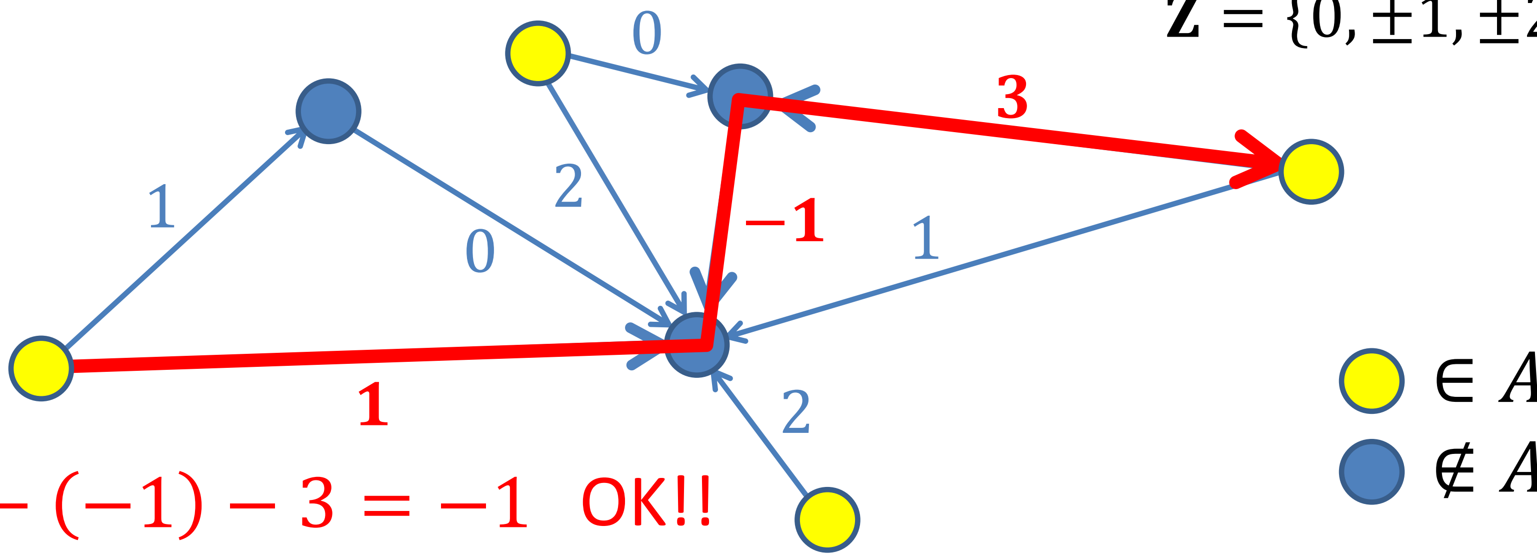

$1 - (-1) - 3 = -1$ OK!!

Find **Maximum Number of**

Fully Vertex-Disjoint Non-zero A-paths

Packing Non-zero A-paths

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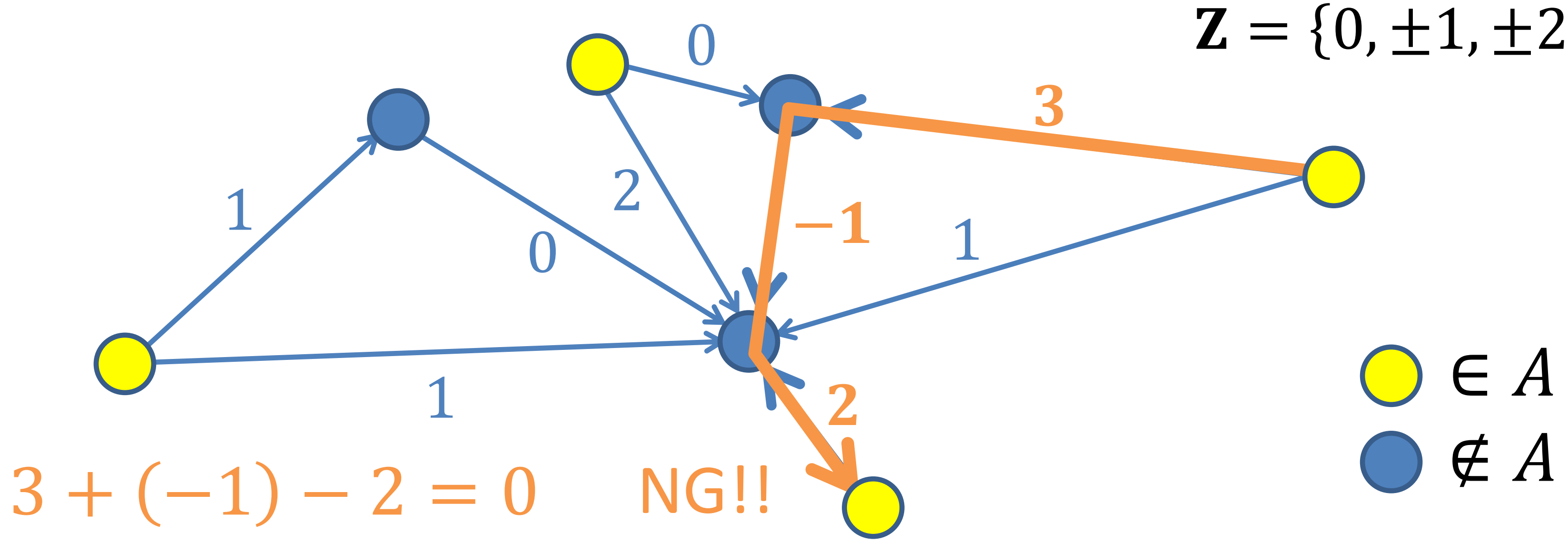

<u>Find</u> **Maximum Number of**

<u>Fully Vertex-Disjoint</u> <u>Non-zero</u> A-paths

Packing Non-zero A-paths

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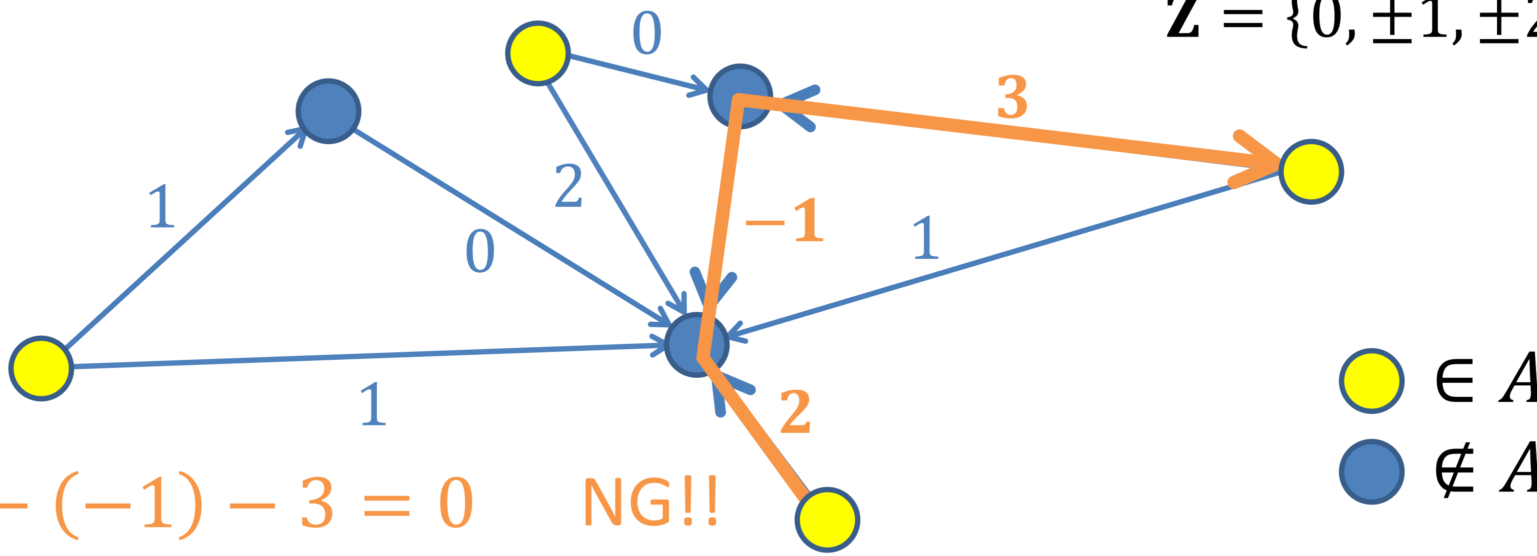

$2 - (-1) - 3 = 0$ NG!!

<u>Find</u> **Maximum Number of**

<u>Fully Vertex-Disjoint</u> <u>Non-zero</u> A-paths

Packing Non-zero A-paths

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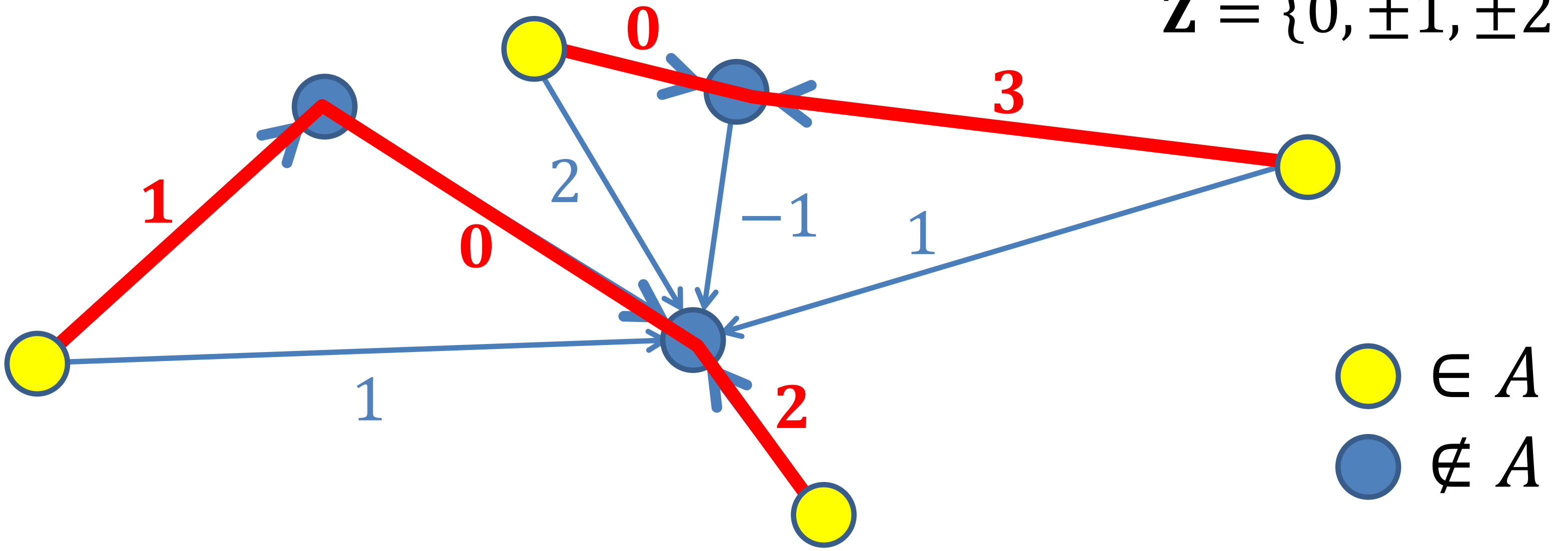


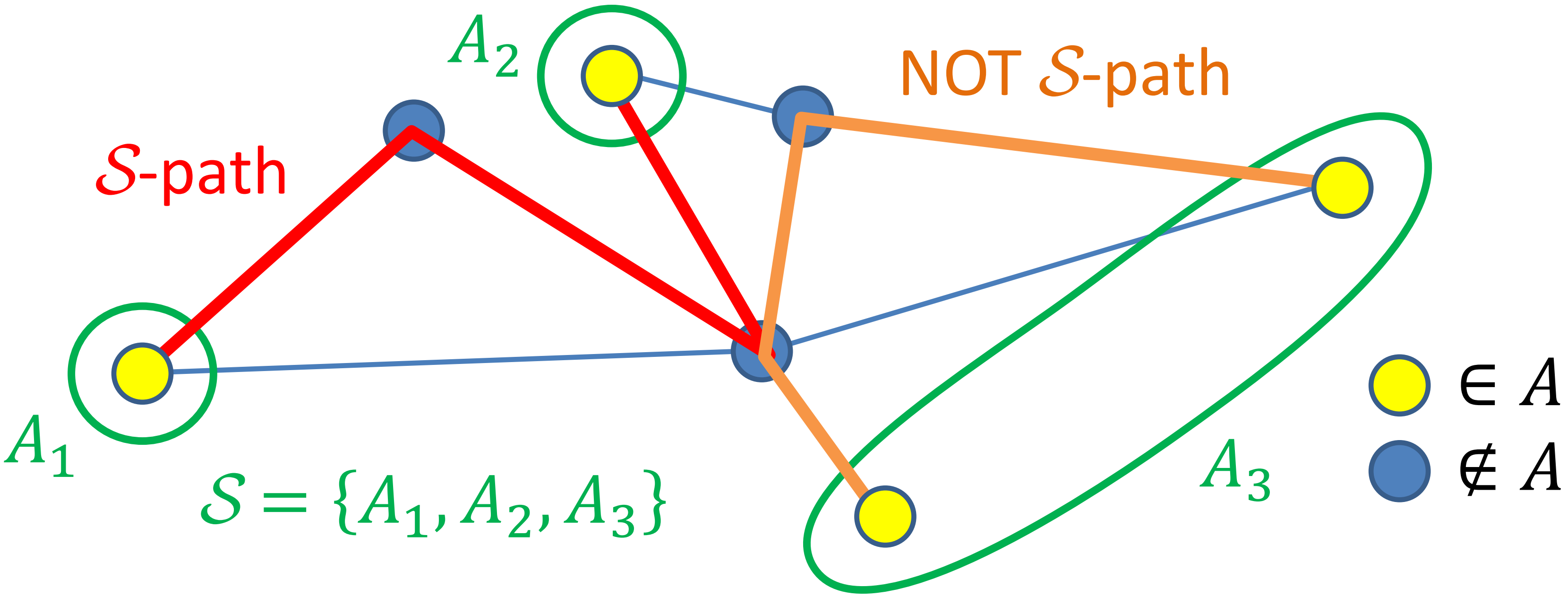
Find **Maximum Number of**

Fully Vertex-Disjoint Non-zero A-paths

Ex. 1 Mader's $\mathcal{S}$-paths

Given $G = (V, E)$: Undirected Graph

$A \subseteq V$: Terminal Set, $\mathcal{S}$: Partition of A



Find **Maximum Number of**
Fully Vertex-Disjoint $\mathcal{S}$-paths

Ex. 1 Mader's $\mathcal{S}$-paths

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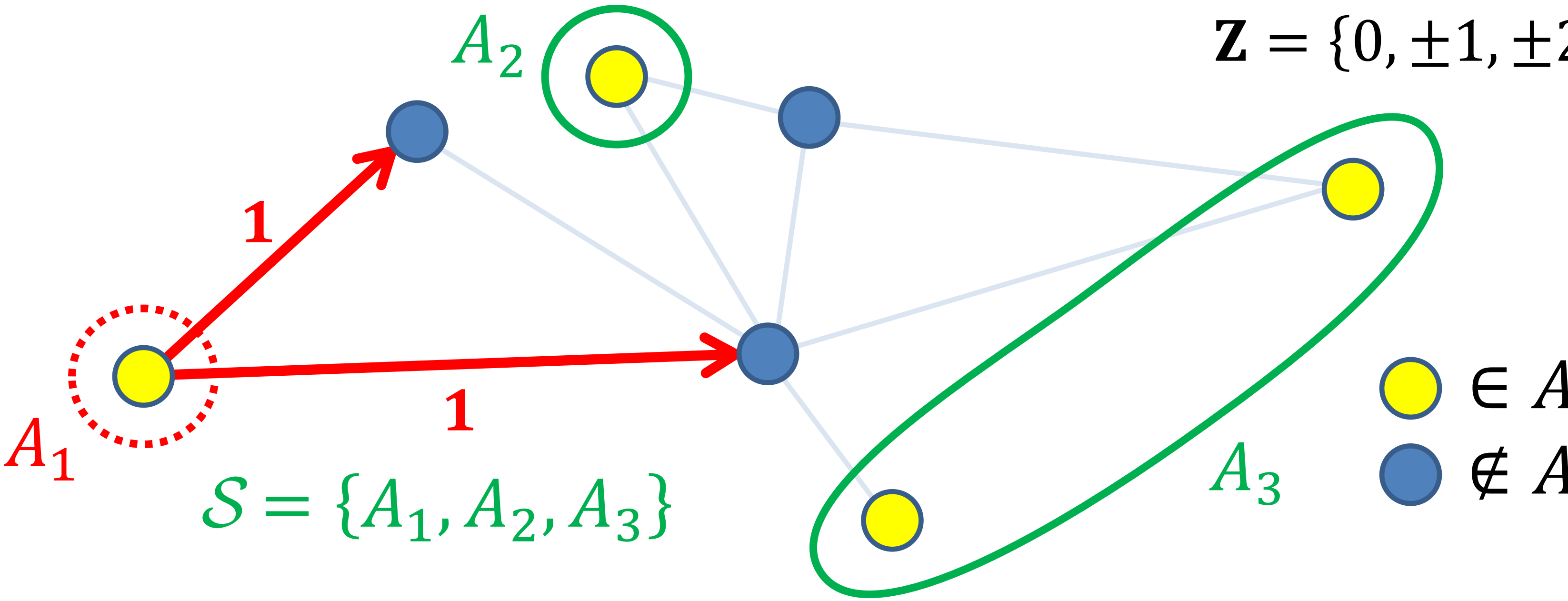
$\mathbf{Z}$-Labeled Graph
$\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$



Find **Maximum Number of**
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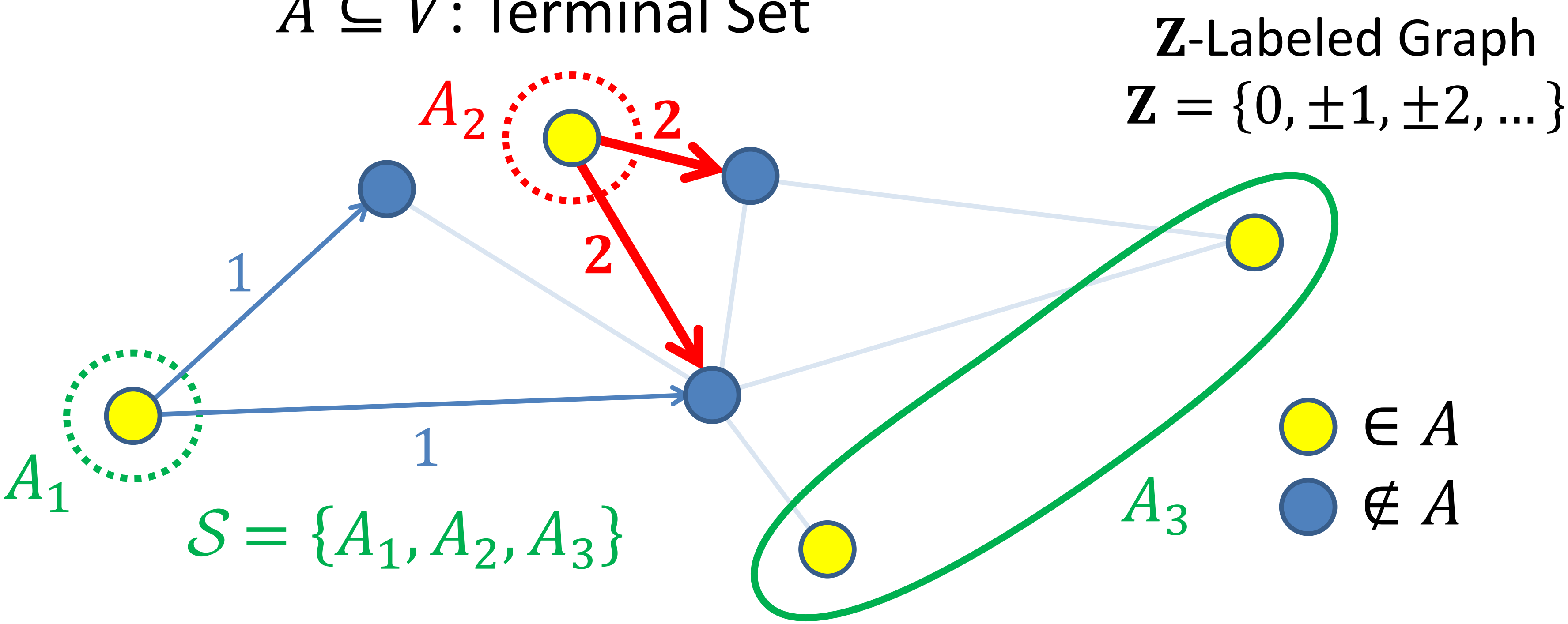

Find **Maximum Number of**
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Ex. 1 Mader's $\mathcal{S}$-paths

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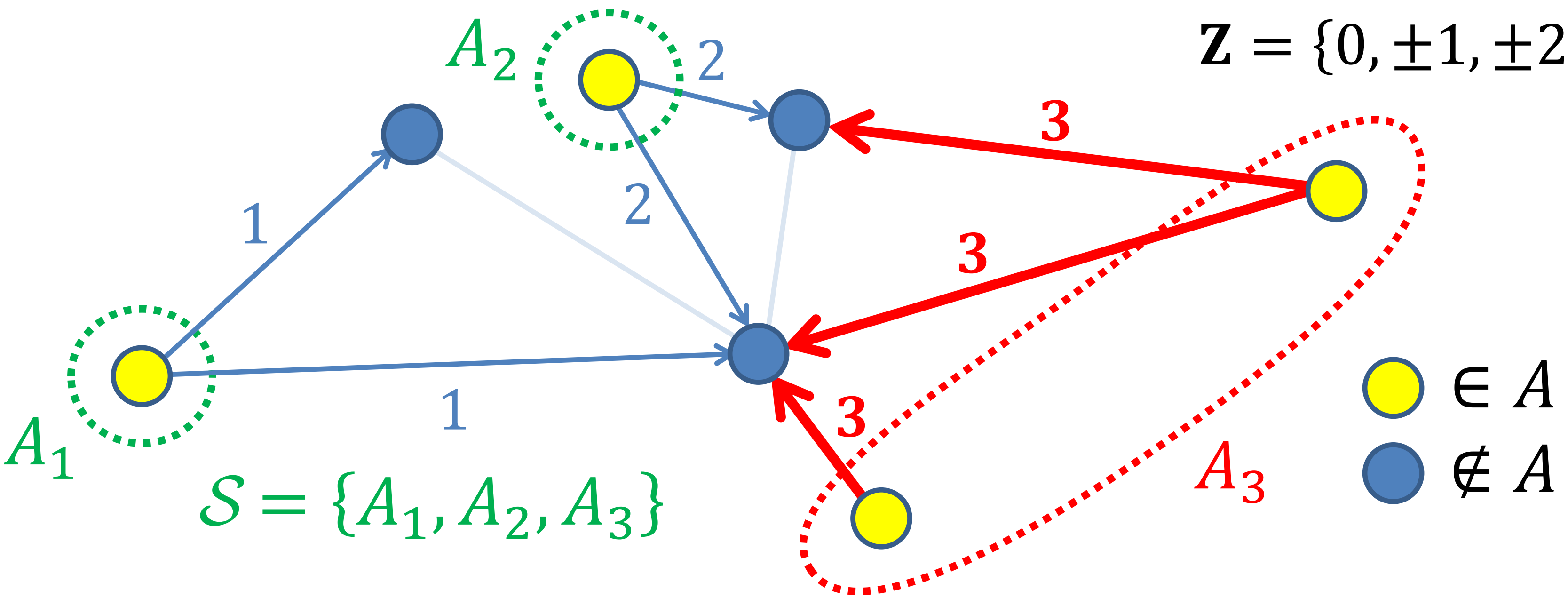

$\mathcal{S} = \{A_1, A_2, A_3\}$

Find **Maximum Number of**

Fully Vertex-Disjoint Non-zero A-paths

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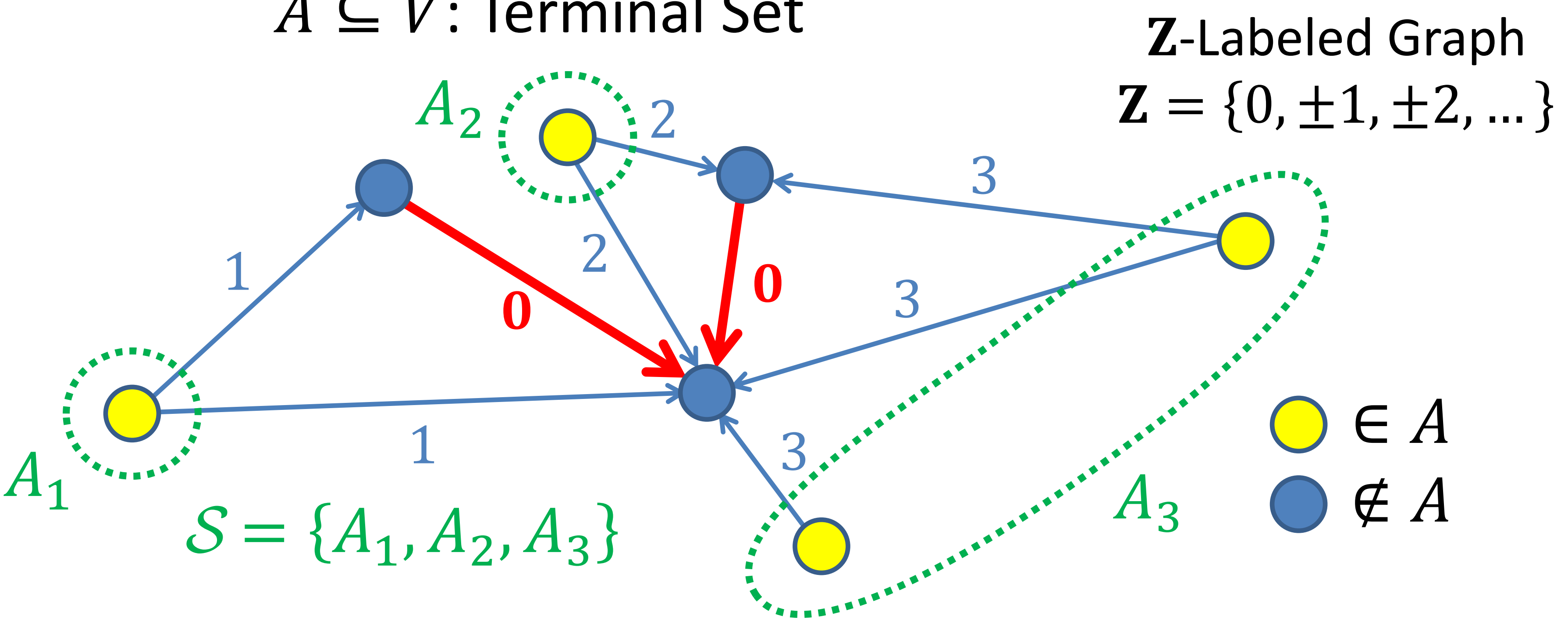

Find **Maximum Number of**

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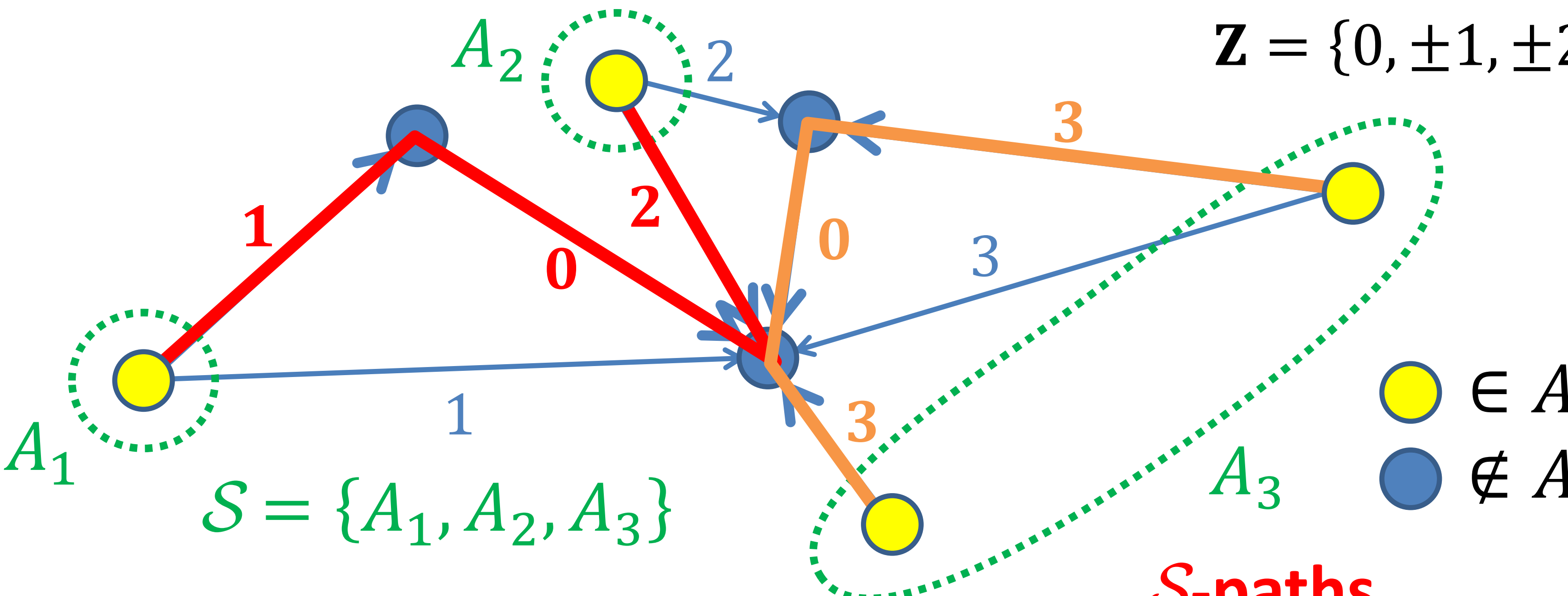


Find **Maximum Number of**
Fully Vertex-Disjoint Non-zero A-paths

$\mathcal{S}$-paths
$\Updownarrow$
Non-zero A-paths

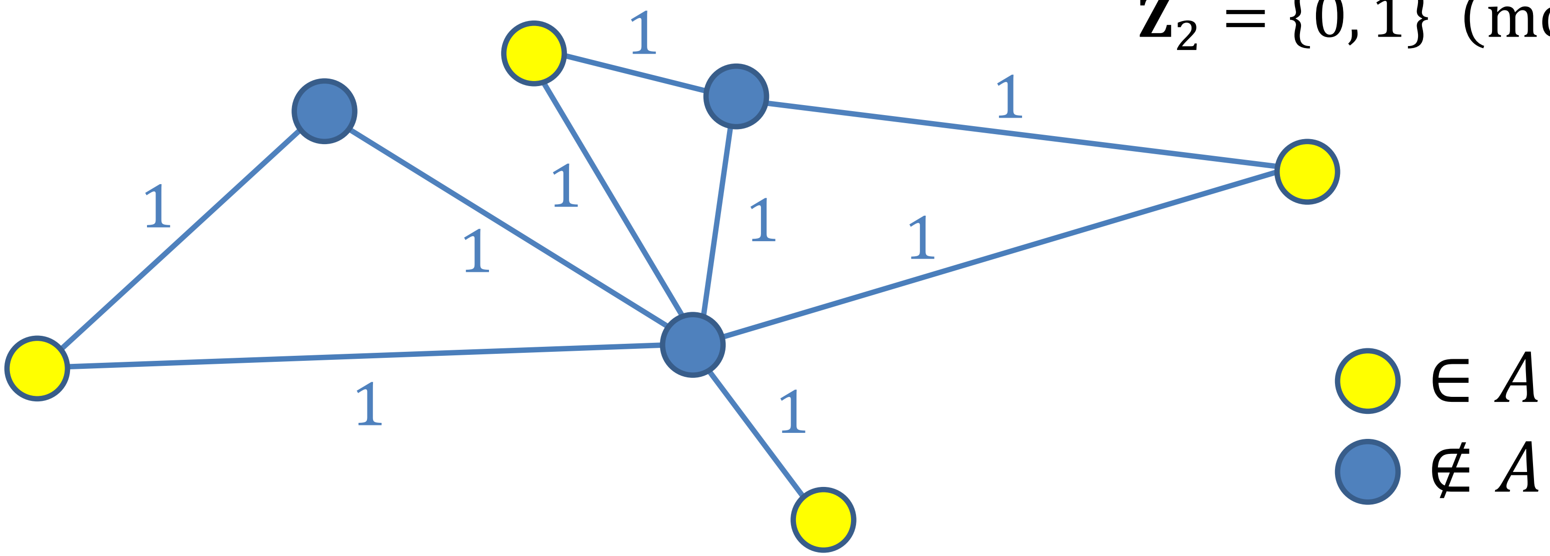
Ex. 2 Odd-Length A-paths

Given $G = (V, E)$: Group-Labeled Graph

$A \subseteq V$: Terminal Set

$\mathbf{Z}_2$-Labeled Graph

$\mathbf{Z}_2 = \{0, 1\} \pmod 2$



Find **Maximum Number of**

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Ex. 2 Odd-Length A-paths

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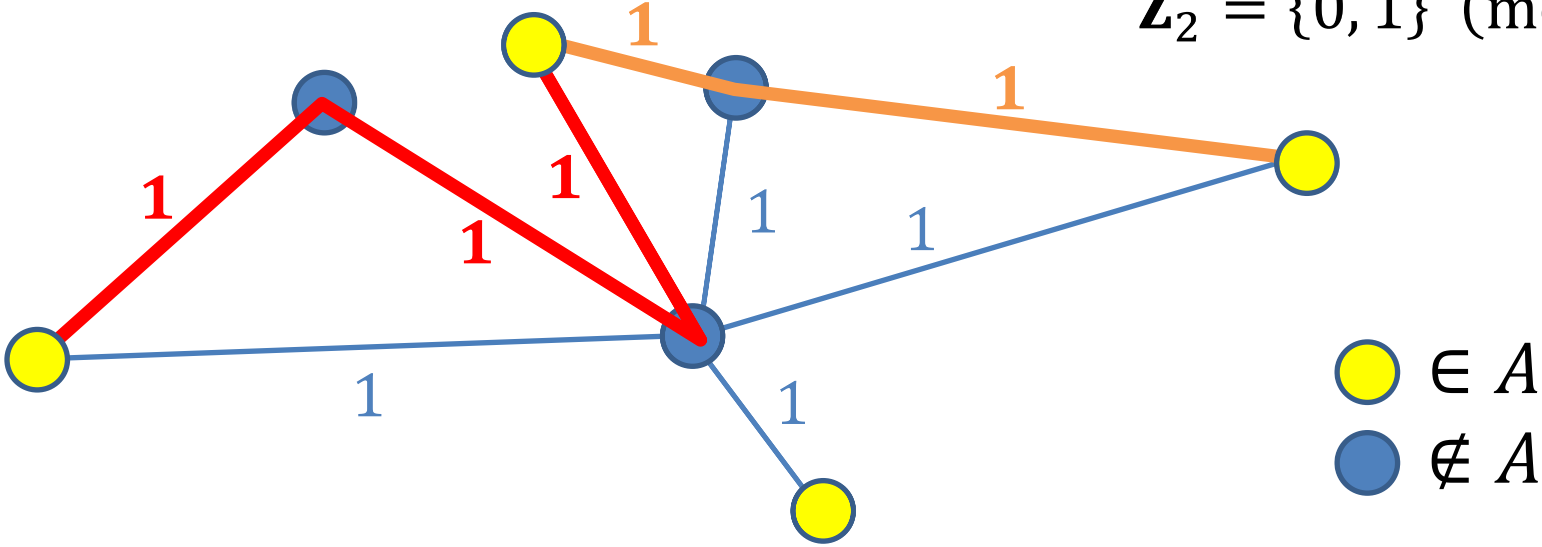
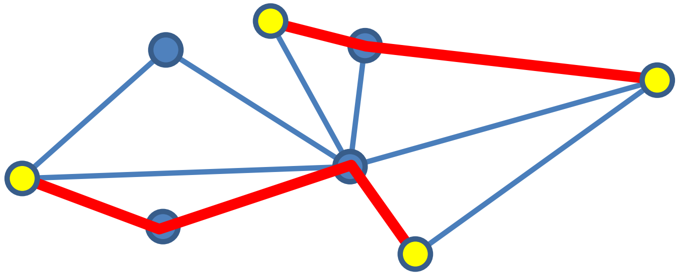


Find **Maximum Number of Fully Vertex-Disjoint Non-zero A-paths**

Odd-Length ⇗⇙ **Non-zero**

Packing Non-zero A-paths

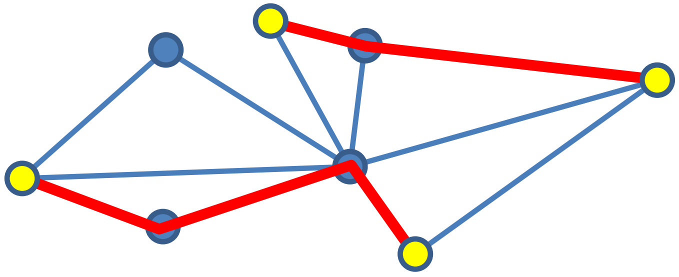
<u>Find</u> **Maximum Number of <u>Fully Vertex-Disjoint</u> Non-zero A-paths**



- Min-Max Formula [Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]
- Polytime Algorithm [Chudnovsky, Cunningham, Geelen 2008]

Packing Non-zero A-paths

<u>Find</u> **Maximum Number of <u>Fully Vertex-Disjoint</u>**
Non-zero A-paths



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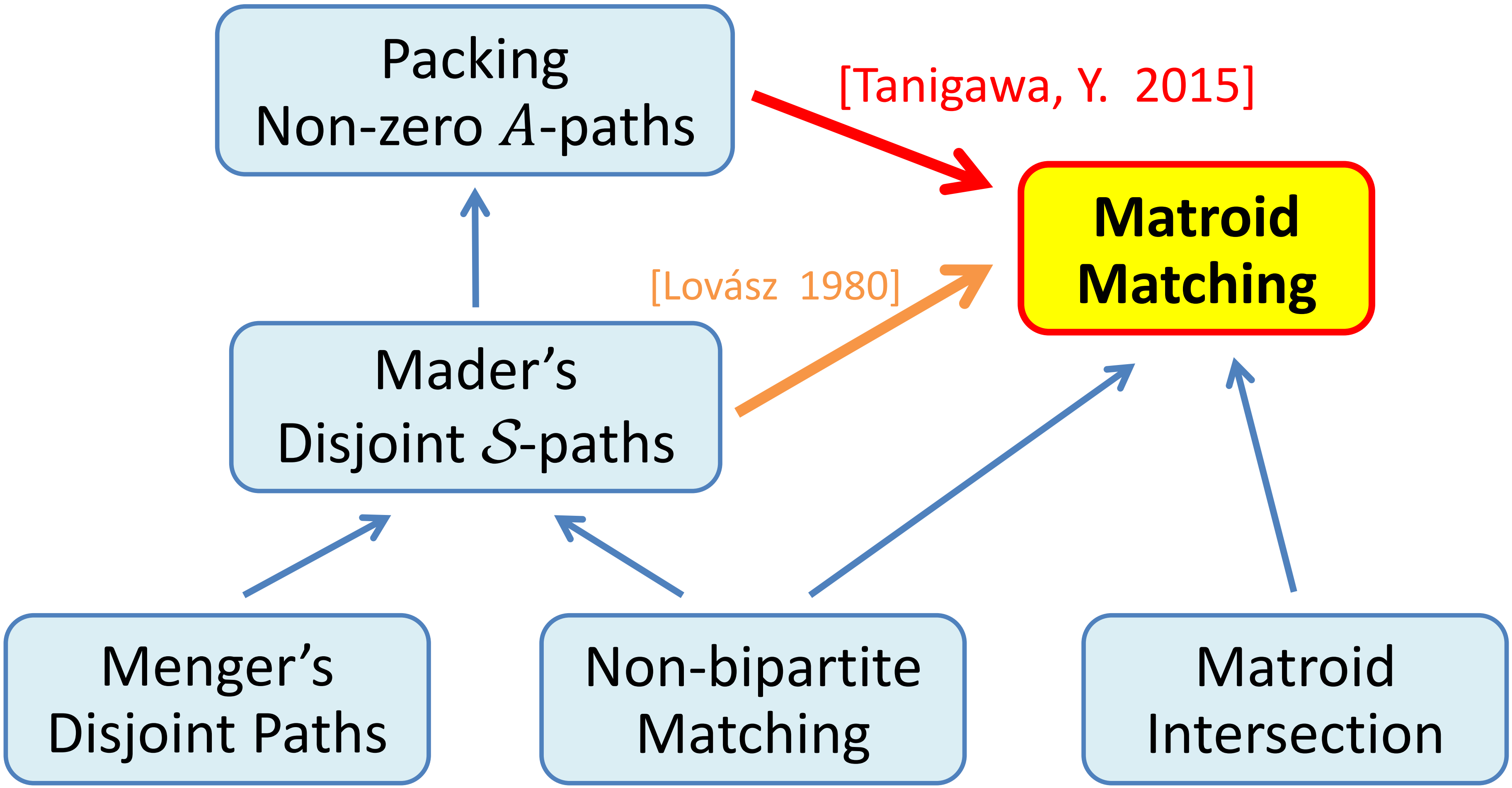
- **Reduction to <u>Matroid Matching</u>** [Tanigawa, Y. 2015]
  → Alternative Proofs for Min-Max and Polytime

Overview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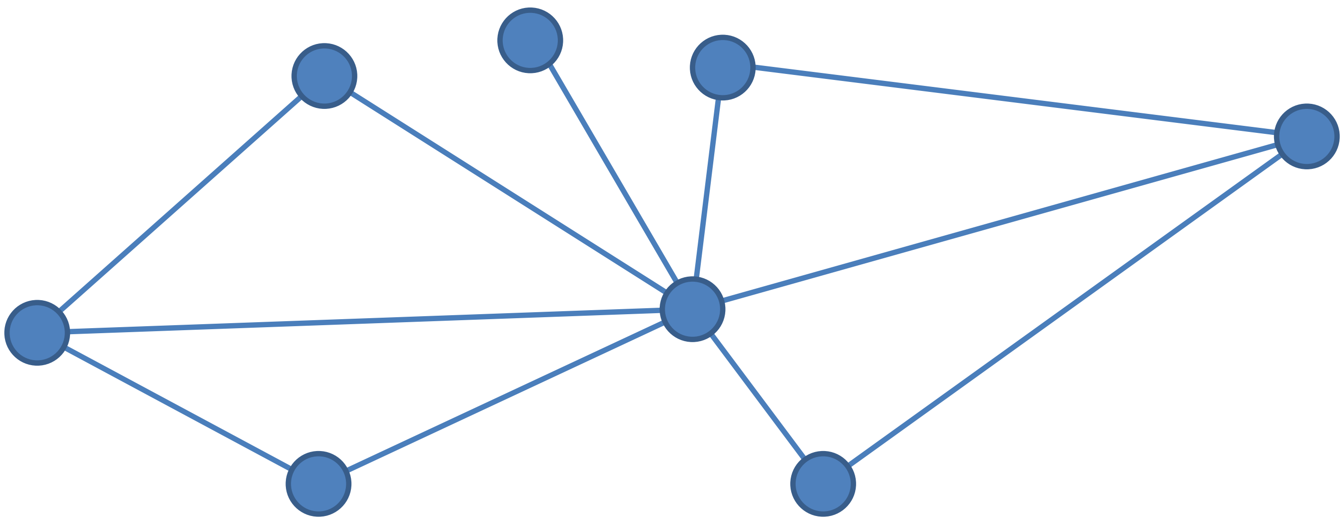
Matroid Matching Problem

Given $G = (V, E)$: Undirected Graph

$\mathbf{M} = (V, \mathcal{I})$: Matroid on Vertex Set



Find **Maximum Matching with Matroid Constraint**

Matroid Matching Problem

<u>Given</u> $G = (V, E)$: Undirected Graph

$\mathbf{M} = (V, \mathcal{I})$: **<u>Matroid</u>** on Vertex Set

$\mathcal{I} \subseteq 2^V$ (Family of **Independent** Sets)

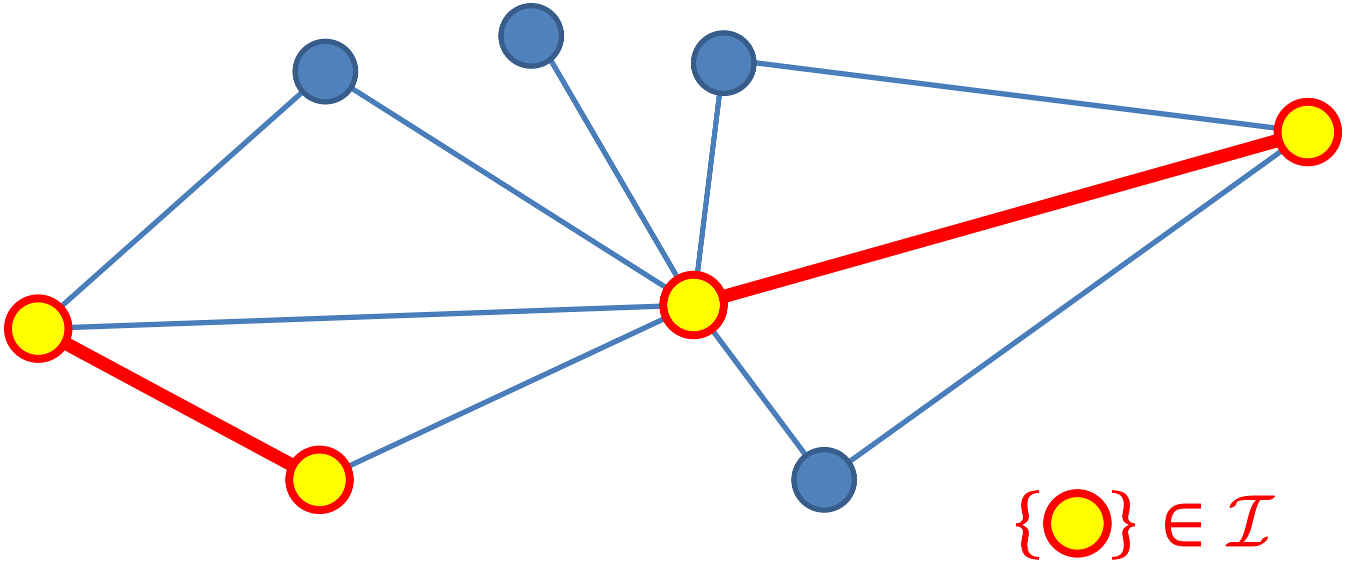
- $\emptyset \in \mathcal{I}$
- $X \subseteq Y \in \mathcal{I} \implies X \in \mathcal{I}$
- $X, Y \in \mathcal{I}$ and $|X| < |Y|$
  $\implies \exists v \in Y \setminus X$ s.t. $X + v \in \mathcal{I}$

<u>Find</u> **Maximum Matching with Matroid Constraint**

Matroid Matching Problem

Given $G = (V, E)$: Undirected Graph

$\mathbf{M} = (V, \mathcal{I})$: Matroid on Vertex Set

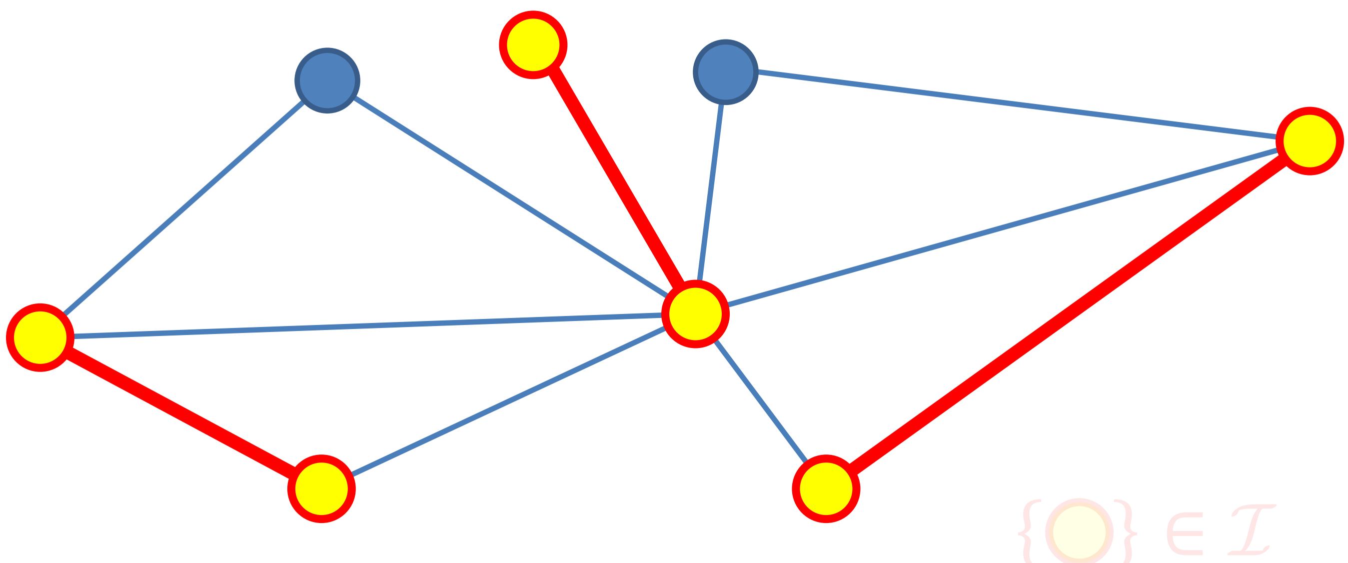


Find **Maximum Matching with Matroid Constraint**

Matroid Matching Problem

Given $G = (V, E)$: Undirected Graph

$\mathbf{M} = (V, \mathcal{I})$: Matroid on Vertex Set



Find **Maximum Matching** with Matroid Constraint

$\mathbf{M} = (V, 2^V)$: Free Matroid $\Longrightarrow$ Maximum Matching

Matroid Matching Problem

Given (S, f): **2-polymatroid**

$\Updownarrow$def

S: Finite Set, $f: 2^S \to \mathbf{Z}$

- $0 \leq f(X) \leq 2|X| \qquad (X \subseteq S)$
- $f(X) \leq f(Y) \qquad (X \subseteq Y \subseteq S)$
- $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (X, Y \subseteq S)$

Find **Maximum Matching**

$X \subseteq S$ with $f(X) = 2|X|$

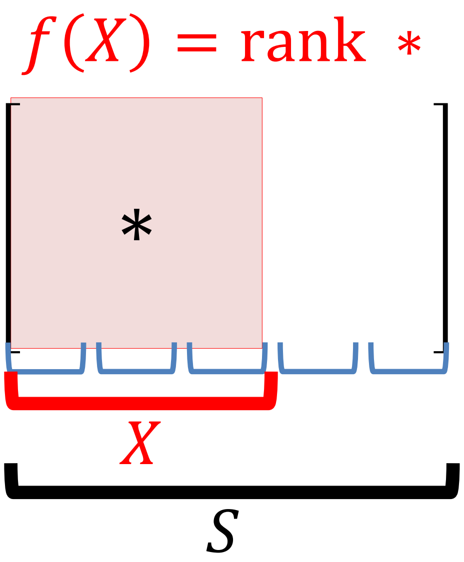
Matroid Matching Problem

Given	(S, f): 2-polymatroid
Find	**Maximum Matching**

- In General, **NOT** Polytime Solvable

Matroid Matching Problem

Given	(S, f): 2-polymatroid
Find	**Maximum Matching**



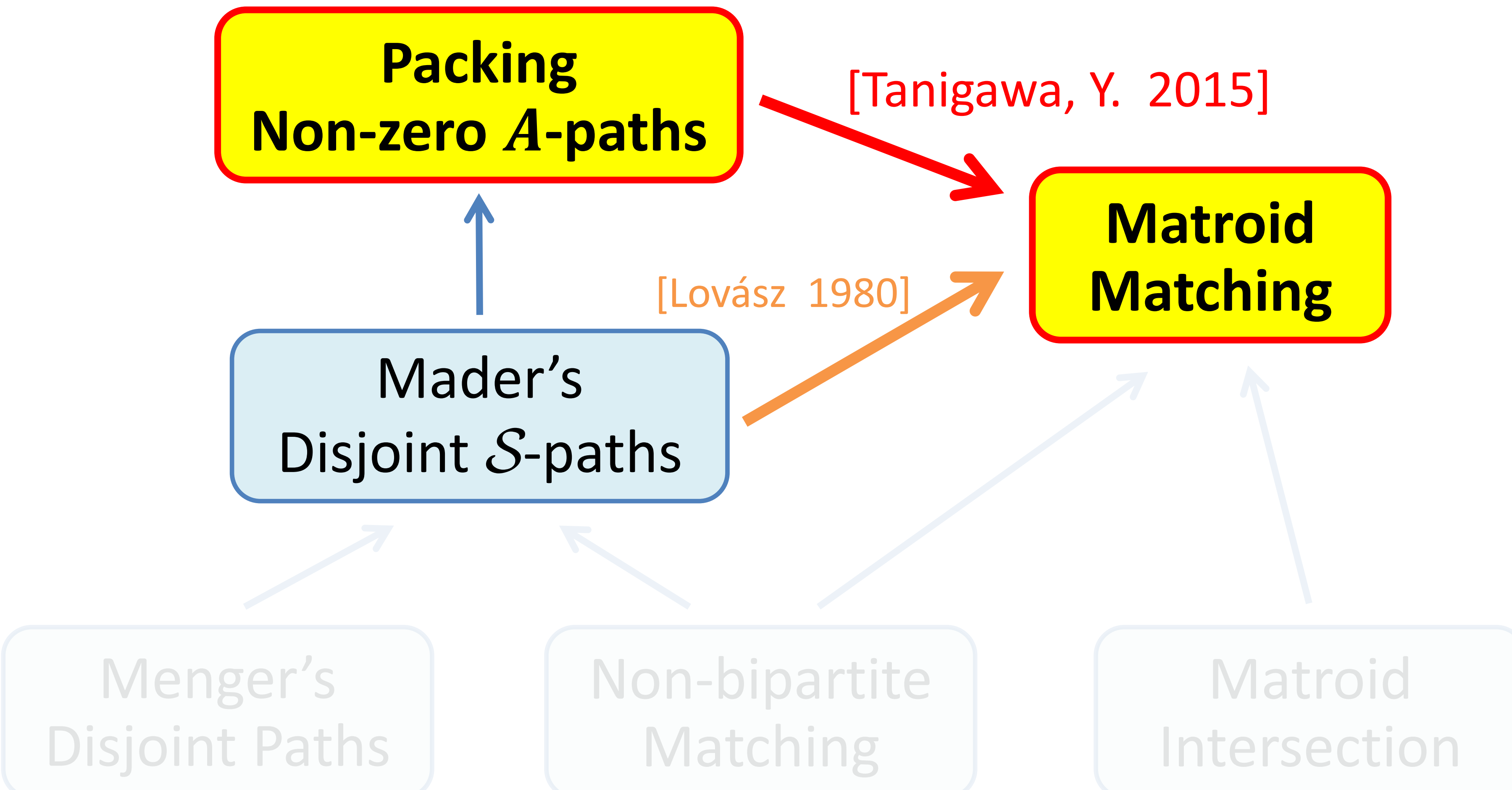
- In General, **NOT** Polytime Solvable
- In **Linear** Case (or More General Case)
  - Min-Max Formula [Lovász 1980]
  - **Polytime** Solvable

$O(m^{17})$? [Lovász 1981]

$O(mn^3)$ (Combinatorial) [Gabow, Stallmann 1986]

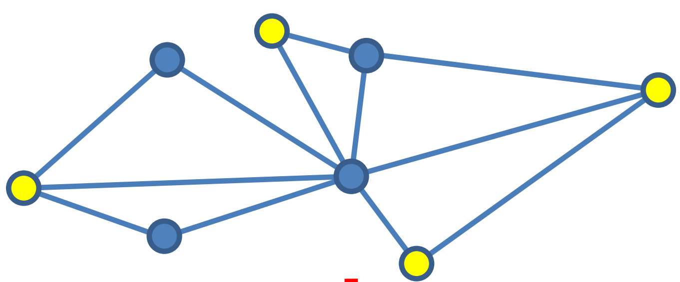
$O(mn^2)$ (Algebraic, w.h.p.) [Cheung, Law, Leung 2011]

Overview

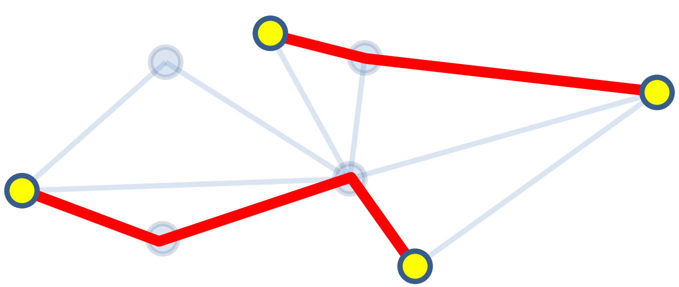

Packing
Non-zero A-paths
[Tanigawa, Y. 2015]
Matroid
Matching
[Lovász 1980]
Mader's
Disjoint $\mathcal{S}$-paths
Menger's
Disjoint Paths
Non-bipartite
Matching
Matroid
Intersection

Reduction Flow

Packing
Non-zero A-paths



Solve



Matroid
Matching

(S, f): 2-polymatroid

Solve

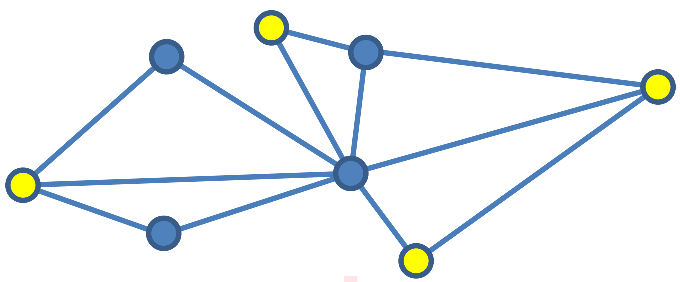
$\underline{X \subseteq S}$
with
$f(X) = 2|X|$

Reduction Flow

Packing
Non-zero A-paths

Reduce →

Matroid
Matching

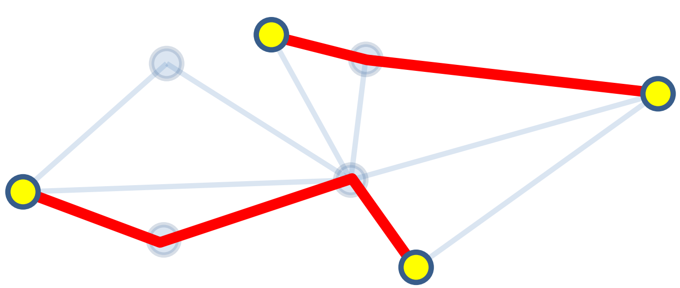


Construct →

(S, f): 2-polymatroid

Solve ↓

Solve

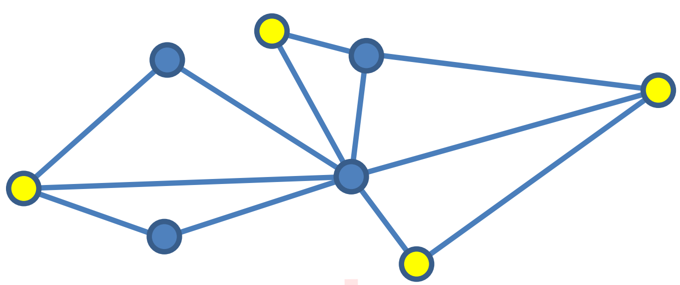


← **Reconstruct**

$\underline{X \subseteq S}$
with
$f(X) = 2|X|$

Reduction Flow

Packing Non-zero A-paths **Reduce** → Matroid Matching

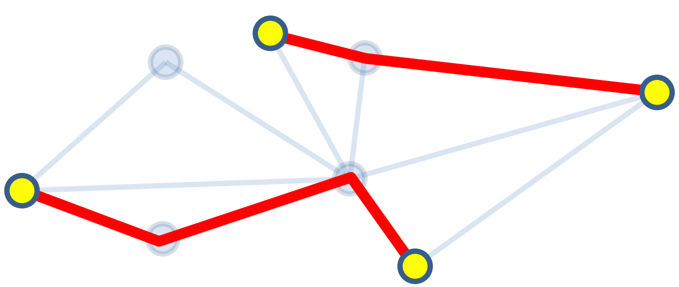


How? → Construct

(S, f): 2-polymatroid

↓ Solve

Solve

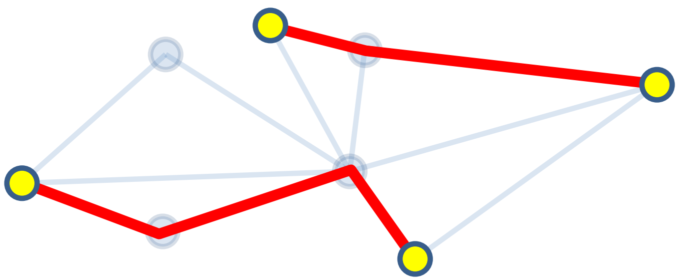


$\underline{X \subseteq S}$ with $f(X) = 2|X|$

How? ← Reconstruct

Our 2-polymatroid

- We want a **Subgraph**



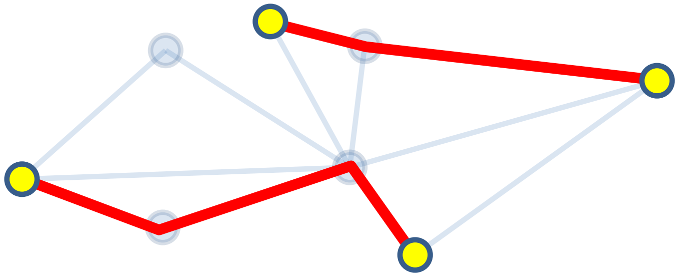
$\underline{X \subseteq S}$

with

$f(X) = 2|X|$

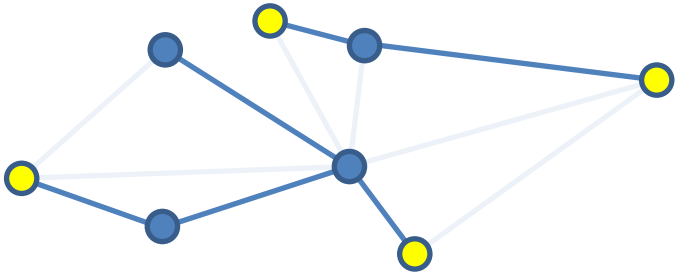
Our 2-polymatroid

- We want a **Subgraph** → $S := E$ (Edge Set)



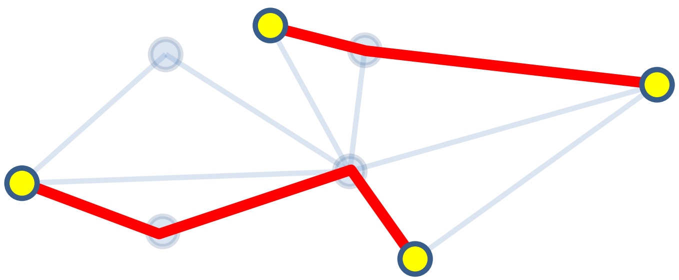
← Reconstruct

$X \subseteq E$
with
$f(X) = 2|X|$



Our 2-polymatroid

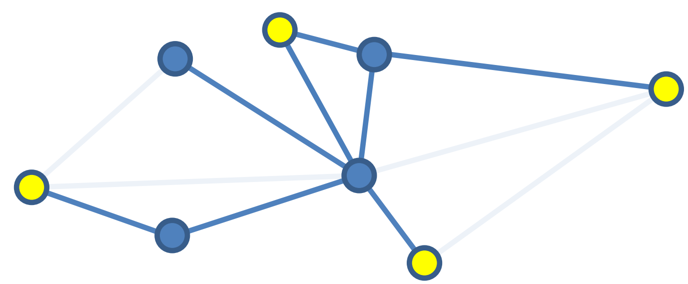
- We want a **Subgraph** $\rightarrow$ $S := E$ (Edge Set)



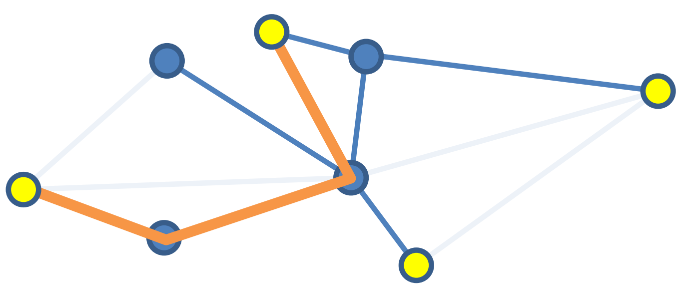
$\longleftarrow$ Reconstruct

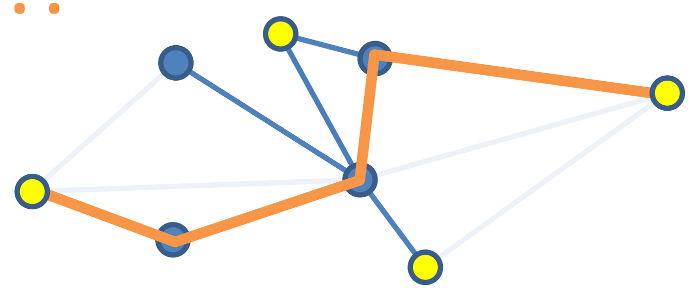
$\underline{X \subseteq E}$
with
$f(X) = 2|X|$

- We want **Easy** Reconstruction



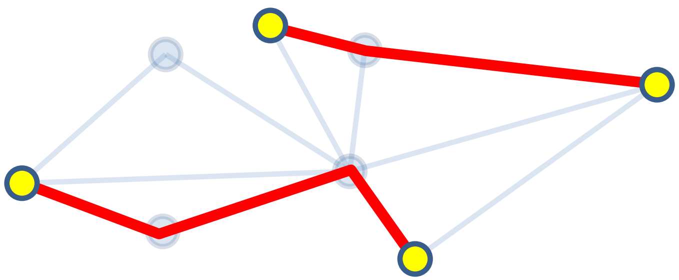
???





Our 2-polymatroid

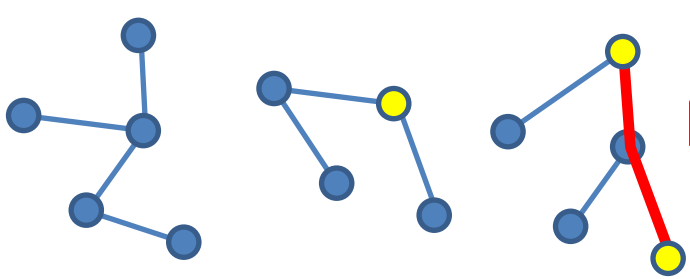
- We want a **Subgraph** → $S := E$ (Edge Set)



← Reconstruct

$\underline{X \subseteq E}$
with
$f(X) = 2|X|$

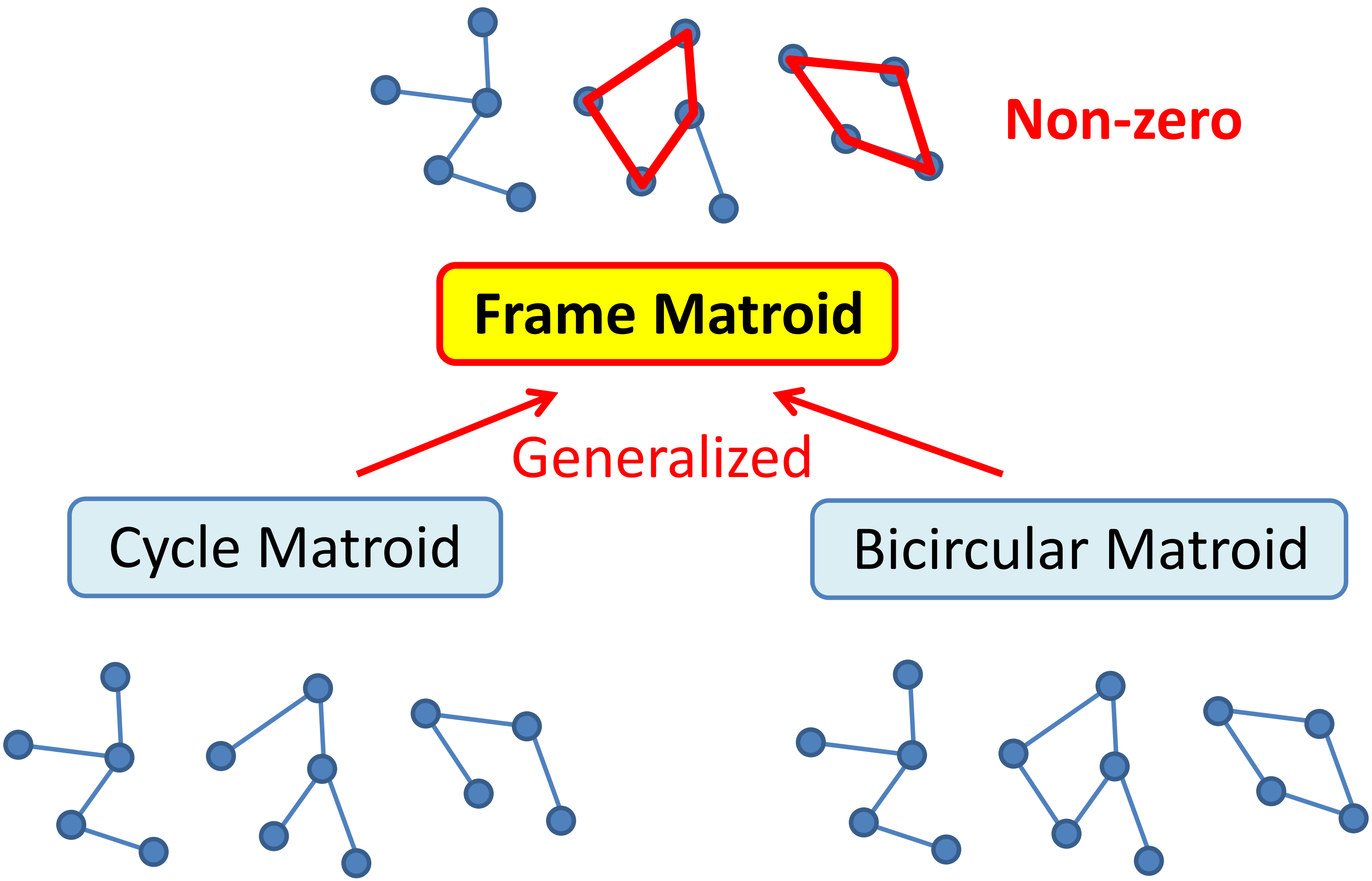
- We want **Easy** Reconstruction



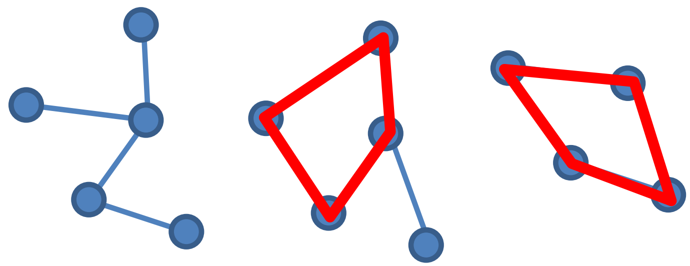
Non-zero $\Leftrightarrow$ $f(X) = 2|X|$

$G[X]$: **Forest**

Key Concept in Our Reduction



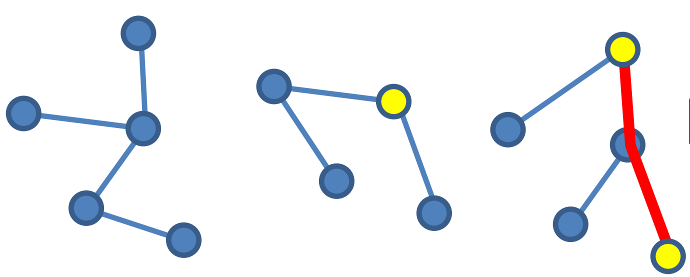
Key Concept in Our Reduction



Non-zero

Frame Matroid

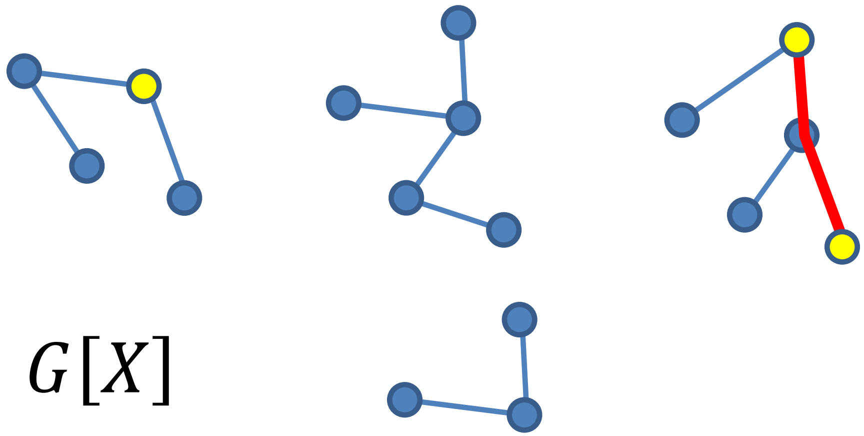
- **Extends** to 2-polymatroid
- **Magic** for Terminals



Non-zero $\iff$ $f(X) = 2|X|$

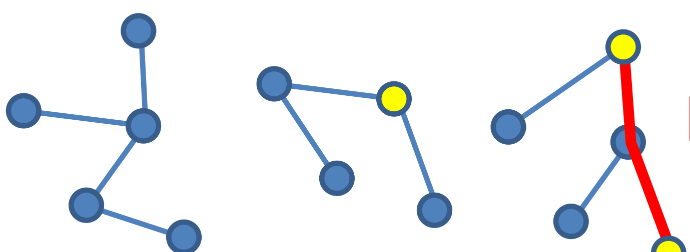
$G[X]$: Forest

Maximal Matching



$G[X]$

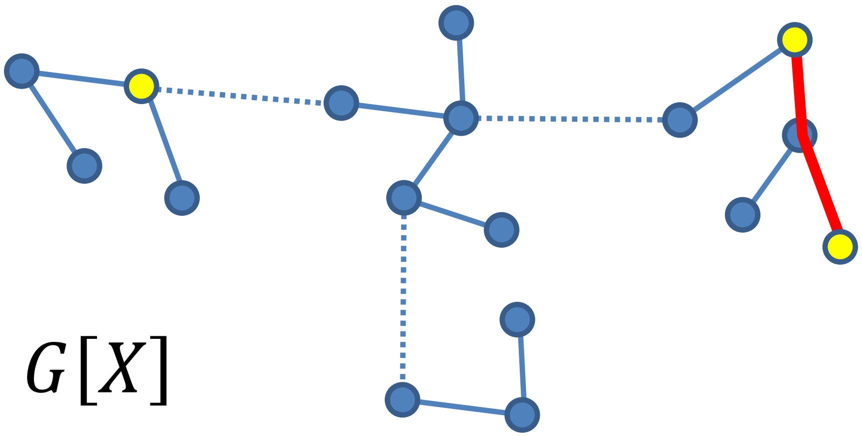
$f(X) = 2|X|$



Non-zero $\Longleftrightarrow$ $f(X) = 2|X|$

$G[X]$: Forest

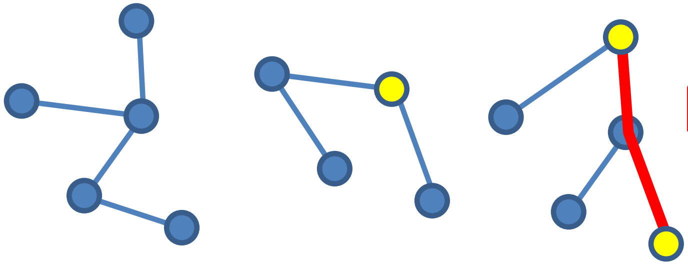
Maximal Matching



$G[X]$

Assume
G: **Connected**

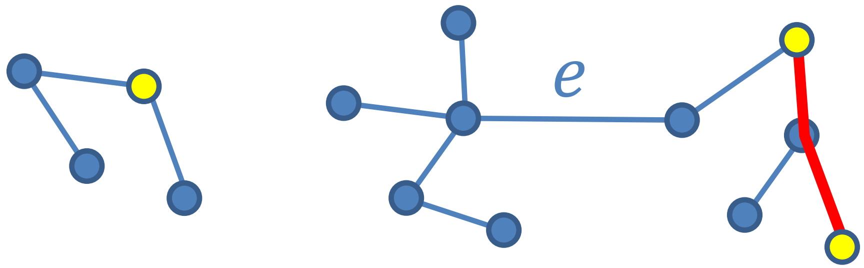
$f(X) = 2|X|$



Non-zero $\Leftrightarrow$ $f(X) = 2|X|$

$G[X]$: Forest

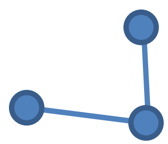
Maximal Matching



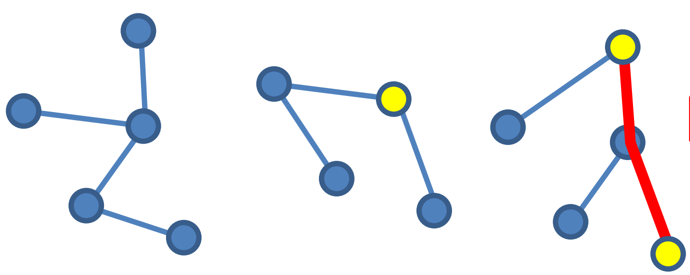
Assume

G: **Connected**

$G[X+e]$



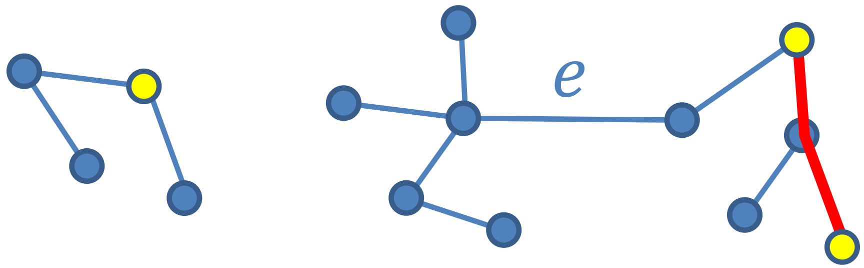
$f(X+e) = 2|X+e|$



Non-zero $\Leftrightarrow$ $f(X) = 2|X|$

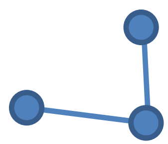
$G[X]$: Forest

Maximal Matching



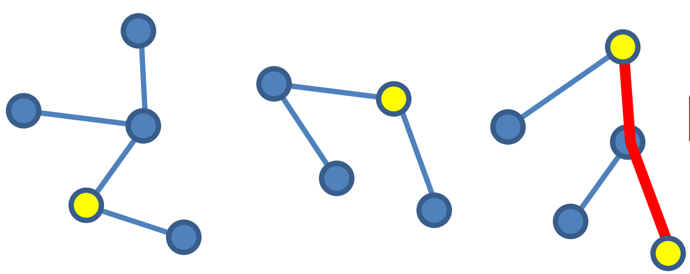
Assume
G: **Connected**

$G[X + e]$



$f(X + e) = 2|X + e|$

G: Connected



Non-zero

$\Longleftrightarrow$

X: **Maximal**
$f(X) = 2|X|$

$G[X]$: Forest

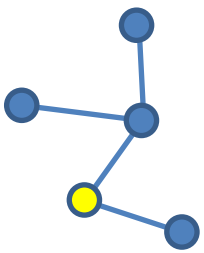
Maximum Matching

X: **Maximal**
$f(X) = 2|X|$ $\Longrightarrow |X| = |V| - \#(\text{Connected Components})$

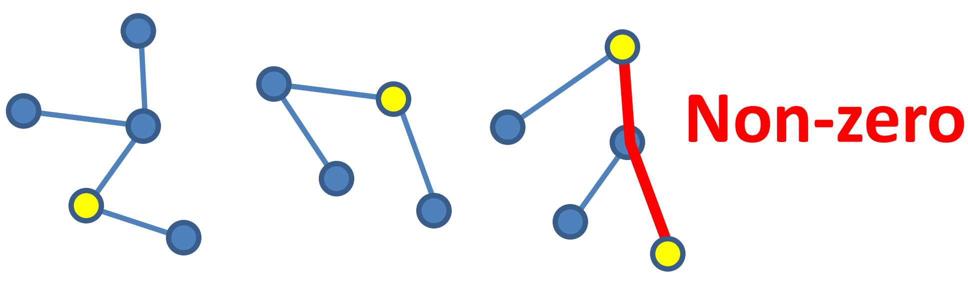
cf. $G' = (V', E')$: Tree

$$\Downarrow$$

$$|E'| = |V'| - 1$$



G: Connected



$G[X]$: Forest

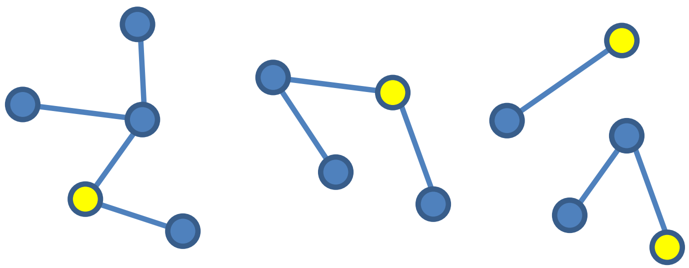
$\Longleftrightarrow$ X: **Maximal**
$f(X) = 2|X|$

Maximum Matching

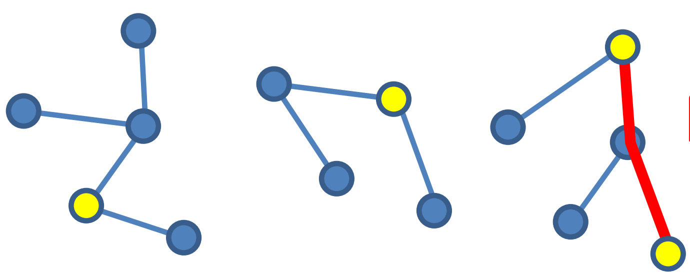
X: **Maximal**
$f(X) = 2|X|$ $\Longrightarrow$ $|X| = |V| -$ #(Connected Components)

||

$|A| -$ #(Non−zero A−paths)



G: Connected



Non-zero

$\Longleftrightarrow$ X: **Maximal**
$f(X) = 2|X|$

$G[X]$: Forest

Maximum Matching

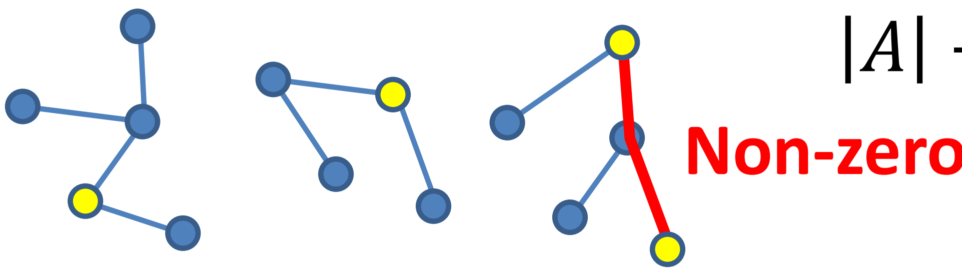
X: **Maximal**

$f(X) = 2|X|$

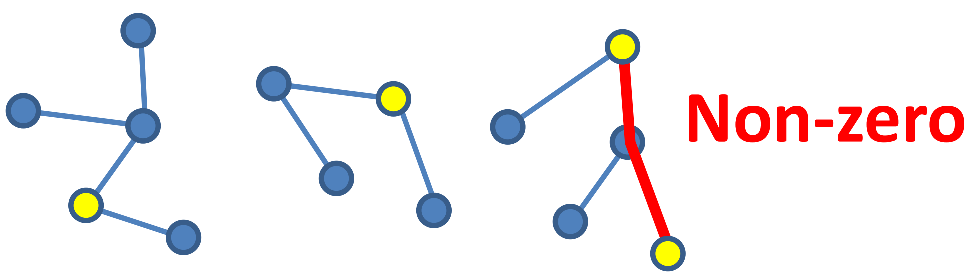
$\Longrightarrow |X| = |V| - \#(\text{Connected Components})$

$\| $

$|A| - \#(\text{Non−zero } A\text{−paths})$



G: Connected



$\Longleftrightarrow$

X: **Maximal**

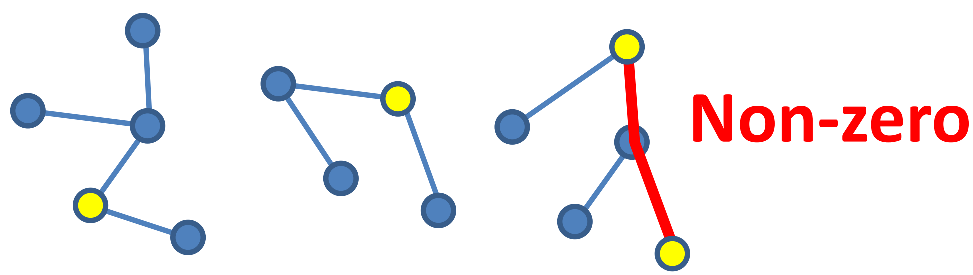
$f(X) = 2|X|$

$G[X]$: Forest

Maximum Matching

X: **Maximal**
$f(X) = 2|X|$ $\Longrightarrow |X| = |V| - |A| + \#(\text{Non–zero } A\text{–paths})$

G: Connected



$\Longleftrightarrow$ X: **Maximal** $f(X) = 2|X|$

$G[X]$: Forest

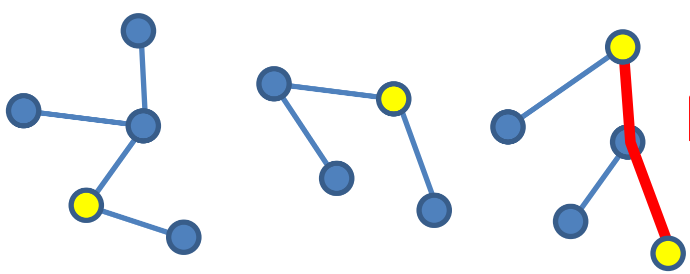
Maximum Matching

X: **Maximal**
$f(X) = 2|X|$ $\Longrightarrow$ $|X| = |V| - |A| + \#(\text{Non-zero } A\text{-paths})$

Fixed ($|V| - |A|$)

$|X|$ → **Maximized** $\Longleftrightarrow$ **Maximized** ← $\#(\text{Non-zero } A\text{-paths})$

G: Connected



Non-zero

$G[X]$: Forest

$\Longleftrightarrow$ X: **Maximal**
$f(X) = 2|X|$

Conclusion

Packing Non-zero A-paths reduces to **Matroid Matching**

- Extends Lovász's Reduction [Lovász 1980]
  of **Mader's $\mathcal{S}$-paths Problem** to **Matroid Matching**

- Alternative Proof for **Min-Max Formula**
  [Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]

- Alternative **Polytime Algorithm** via **Matroid Matching**
  [Lovász 1981]

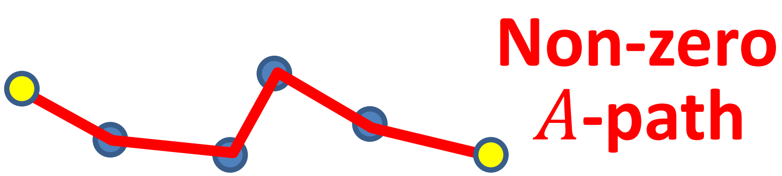
  (cf. Faster Algorithms via **Linear Matroid Parity**
  under **Representability Cond.** for Group) [Y. 2014]

Min-Max Formula for Non-zero A-paths

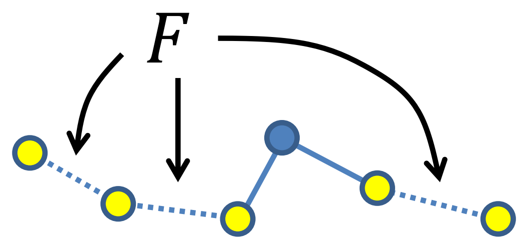
$\mu(G, A) \coloneqq \max \#(\text{Disjoint Non–zero } A\text{–paths in } G)$
$\hat{\mu}(G, A) \coloneqq \max \#(\text{Disjoint } A\text{–paths in } G)$

$$\mu(G, A) \leq \hat{\mu}(G - F, A_F)$$

- $F \subseteq E(G)$ Contains **NO** Non-zero A-paths
- $A_F \coloneqq A \cup V(F)$



G, A



$G - F, A_F$

Min-Max Formula for Non-zero A-paths

$\mu(G, A) \coloneqq \max \#(\text{Disjoint Non−zero } A\text{−paths in } G)$

$\hat{\mu}(G, A) \coloneqq \max \#(\text{Disjoint } A\text{−paths in } G)$

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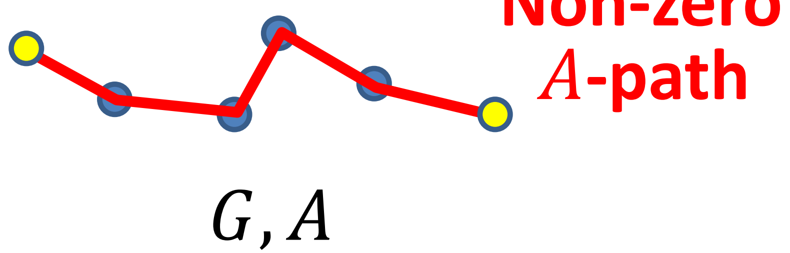
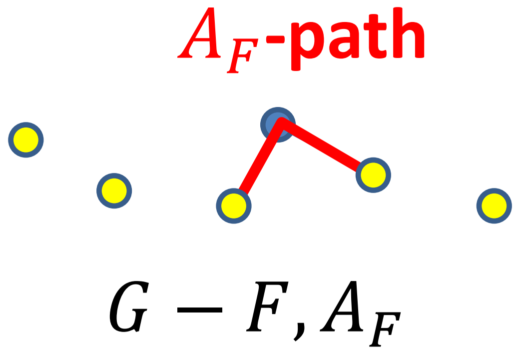




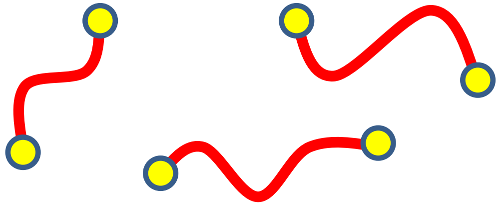
Min-Max Formula for Non-zero A-paths

$\mu(G, A) \coloneqq \max \#(\text{Disjoint Non–zero } A\text{–paths in } G)$

$\hat{\mu}(G, A) \coloneqq \max \#(\text{Disjoint } A\text{–paths in } G)$

$$\mu(G, A) \leq \hat{\mu}(G - F, A_F)$$
$$\leq \hat{\mu}(G - F - X, A_F \setminus X) + |X|$$

- $F \subseteq E(G)$ Contains **NO** Non-zero A-paths
- $A_F \coloneqq A \cup V(F)$
- $X \subseteq V(G)$



$G - F, A_F$

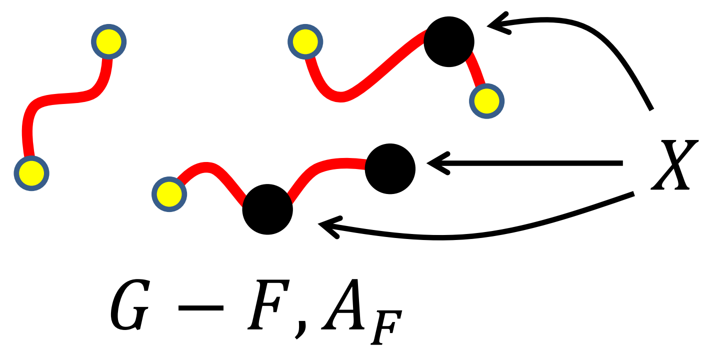
Min-Max Formula for Non-zero A-paths

$\mu(G,A) :=$ max #(Disjoint Non−zero A−paths in G)

$\hat{\mu}(G,A) :=$ max #(Disjoint A−paths in G)

$$\mu(G,A) \leq \hat{\mu}(G-F, A_F)$$
$$\leq \hat{\mu}(G-F-X, A_F \setminus X) + |X|$$

- $F \subseteq E(G)$ Contains **NO** Non-zero A-paths
- $A_F := A \cup V(F)$
- $X \subseteq V(G)$



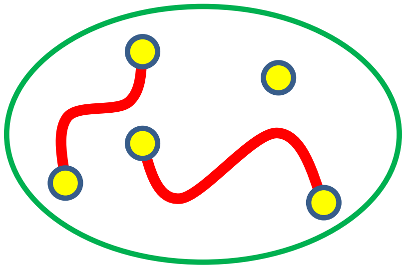
Min-Max Formula for Non-zero A-paths

$\mu(G, A) \coloneqq \max \#(\text{Disjoint Non–zero } A\text{–paths in } G)$

$\hat{\mu}(G, A) \coloneqq \max \#(\text{Disjoint } A\text{–paths in } G)$

$$\mu(G,A) \le \hat{\mu}(G - F, A_F)$$
$$\le \hat{\mu}(G - F - X, A_F \setminus X) + |X|$$
$$\le \sum_{H \in \mathrm{comp}(G-F-X)} \left\lfloor \frac{|V(H) \cap A_F|}{2} \right\rfloor + |X|$$

- $F \subseteq E(G)$ Contains **NO** Non-zero A-paths
- $A_F \coloneqq A \cup V(F)$
- $X \subseteq V(G)$



H: Conn. Comp.

Min-Max Formula for Non-zero A-paths

$$\mu(G, A) \coloneqq \max \#(\text{Disjoint Non–zero } A\text{–paths in } G)$$

<u>Thm</u>.

$$\mu(G, A) = \min_{F, X} \sum_{H \in \text{comp}(G - F - X)} \left\lfloor \frac{|V(H) \cap A_F|}{2} \right\rfloor + |X|$$

- $F \subseteq E(G)$ Contains **<u>NO</u>** Non-zero A-paths
- $A_F \coloneqq A \cup V(F)$
- $X \subseteq V(G)$

[Chudnovsky, Geelen, Gerards, Goddyn, Lohman, Seymour 2006]