Making Bipartite Graphs DM-irreducible

Kristóf Bérczi[1], Satoru Iwata[2], Jun Kato[3], **Yutaro Yamaguchi**[4]

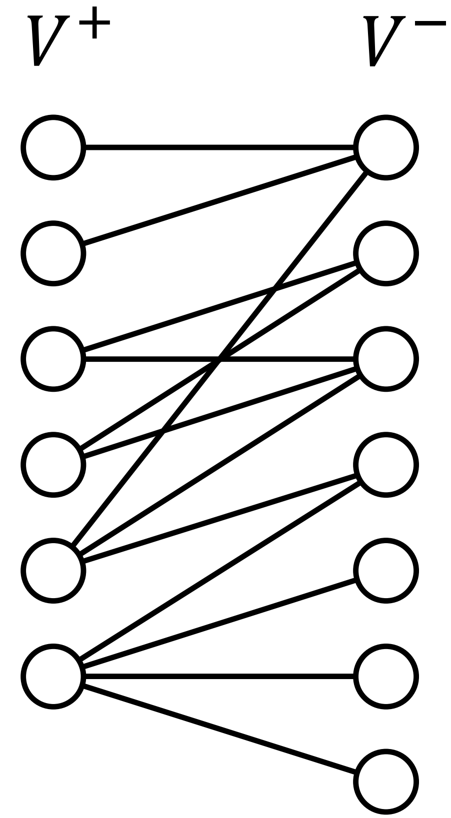
1. Eötvös Lorand University, Hungary.
2. University of Tokyo, Japan.
3. TOYOTA Motor Corporation, Japan.
4. Osaka University, Japan.

ISMP 2018 @Bordeaux July 5, 2018

Dulmage–**M**endelshon Decomposition

[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph

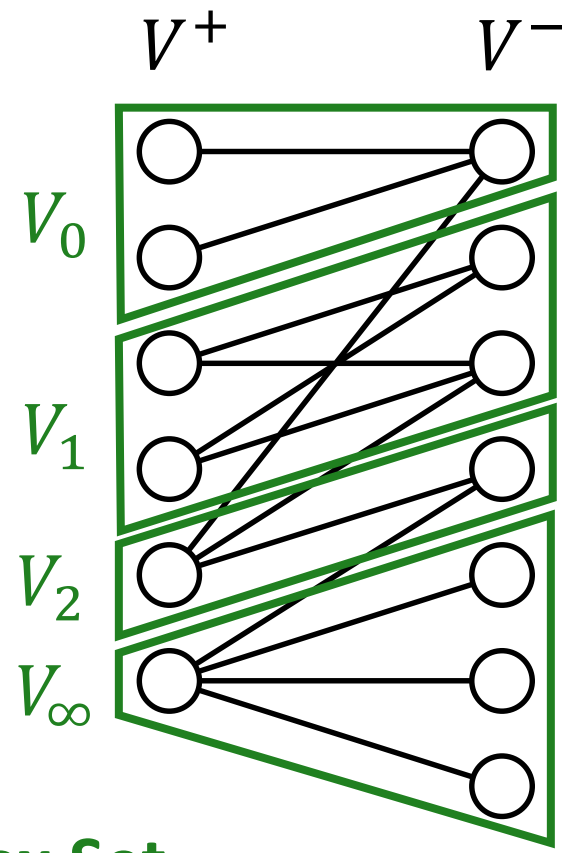


Dulmage–**M**endelshon Decomposition

[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph

- $|V_0^+| > |V_0^-|$ or $V_0 = \emptyset$
- $|V_i^+| = |V_i^-|$ $(i \neq 0, \infty)$
- $|V_\infty^+| < |V_\infty^-|$ or $V_\infty = \emptyset$
- $\forall$**Max. Matching** in G is a union of **Perfect Matchings** in $G[V_i]$



Unique Partition of Vertex Set

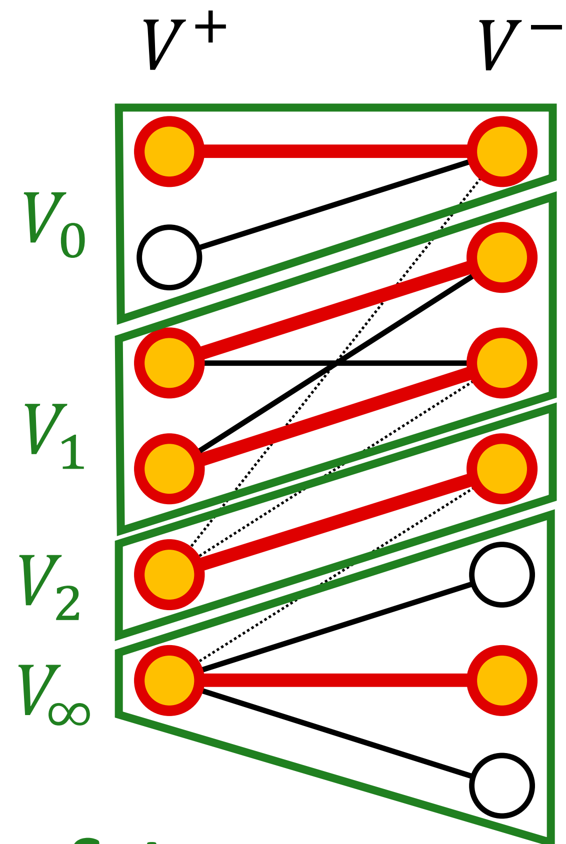
reflecting Structure of **Maximum Matchings**

(Definition will be given later)

Dulmage–Mendelshon Decomposition

[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph



- $|V_0^+| > |V_0^-|$ or $V_0 = \emptyset$
- $|V_i^+| = |V_i^-|$ $(i \neq 0, \infty)$
- $|V_\infty^+| < |V_\infty^-|$ or $V_\infty = \emptyset$

- $\forall$**Max. Matching** in G is a union of **Perfect Matchings** in $G[V_i]$

Covering ALL vertices in one side

Unique Partition of Vertex Set

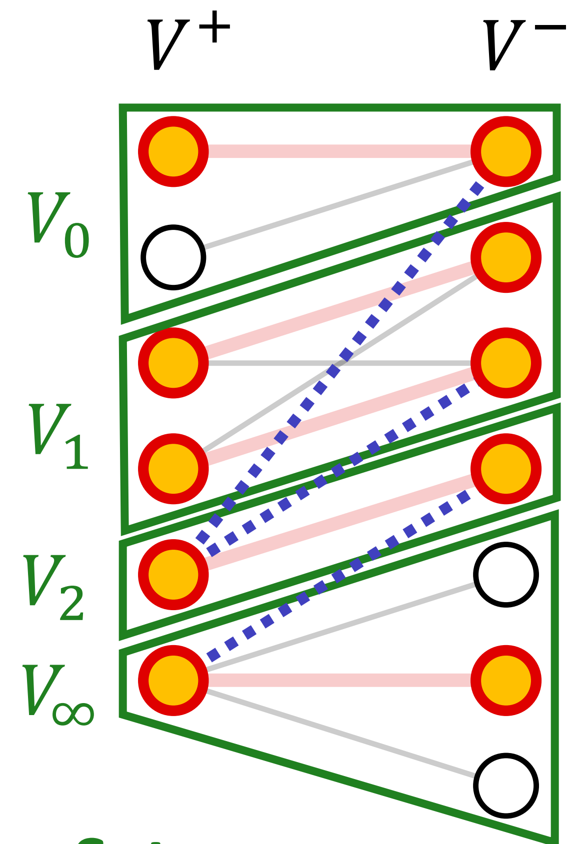
reflecting Structure of **Maximum Matchings**

(Definition will be given later)

Dulmage–Mendelshon Decomposition

[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph



- $\forall$**Max. Matching** in G is a union of **Perfect Matchings** in $G[V_i]$

  → **Edges** between V_i and V_j $(i \neq j)$ can**NOT** be used.

- $\forall e$: Edge in $G[V_i]$, $\exists$**Perfect Matching** in $G[V_i]$ using e

Unique Partition of Vertex Set

reflecting Structure of **Maximum Matchings**

(Definition will be given later)

Dulmage–**M**endelshon Decomposition

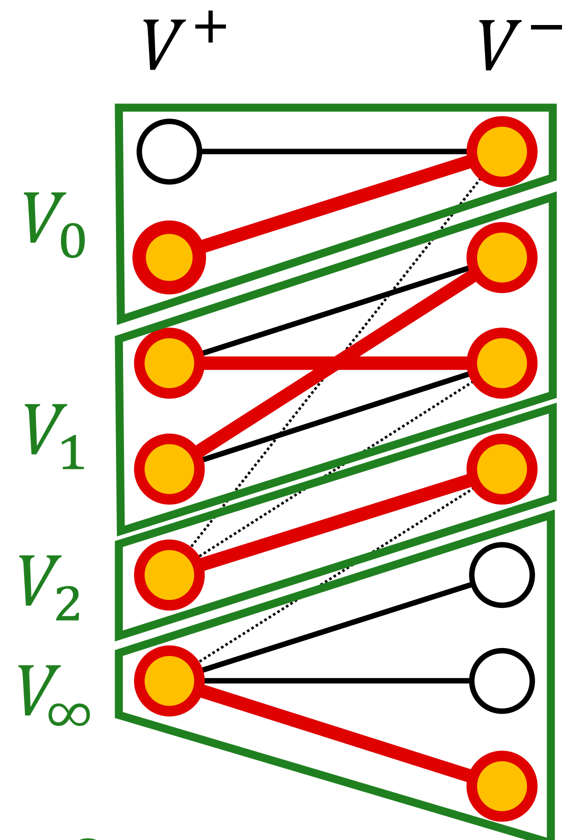
[Dulmage–Mendelsohn 1958,59]

$G = (V^+, V^-; E)$: Bipartite Graph

- $\forall$**Max. Matching** in G is a union of **Perfect Matchings** in $G[V_i]$

  $\rightarrow$ **Edges** between V_i and V_j $(i \neq j)$ can**NOT** be used.

- $\forall e$: Edge in $G[V_i]$, $\exists$**Perfect Matching** in $G[V_i]$ using e



Unique Partition of Vertex Set

reflecting Structure of **Maximum Matchings**

(Definition will be given later)

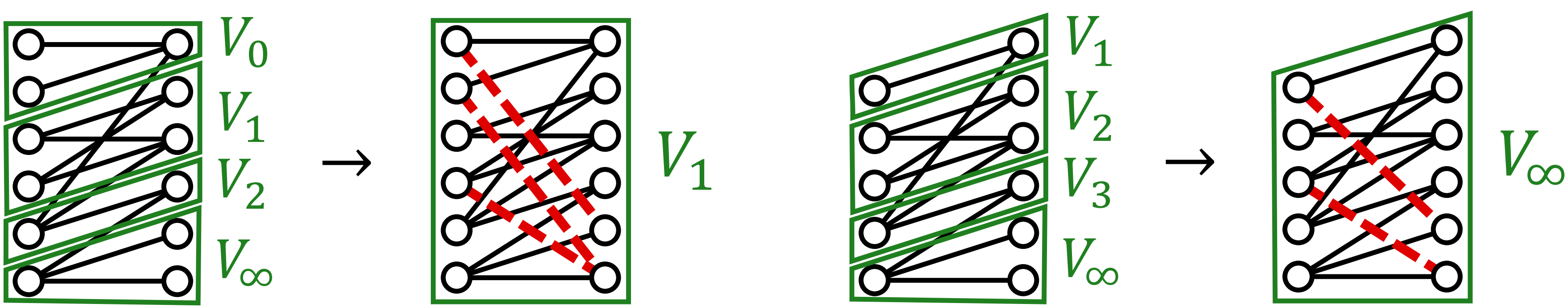
Our Problem

Input $G = (V^+, V^-; E)$: Bipartite Graph

Goal Find a **Minimum Number of Additional Edges** to Make G **DM-irreducible**

DM-decomposition consists of a **Single Component**

$= [\forall e, \exists M$: Perfect Matching s.t. $e \in M] + \alpha$ (Some Connectivity)



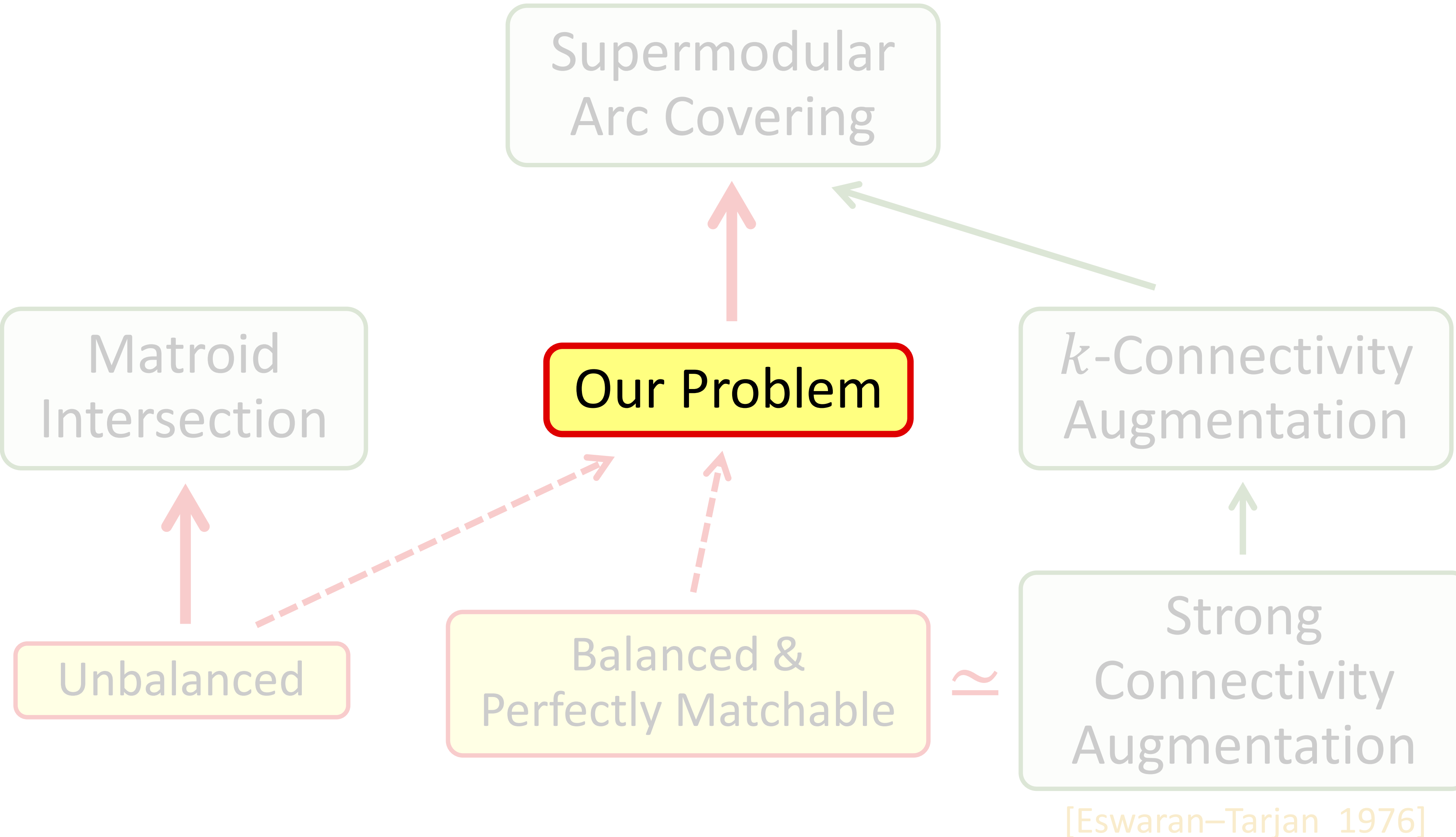
Our Results (Summary)

Input $G = (V^+, V^-; E)$: Bipartite Graph

Goal Find a **Minimum Number of Additional Edges** to Make G **DM-irreducible**

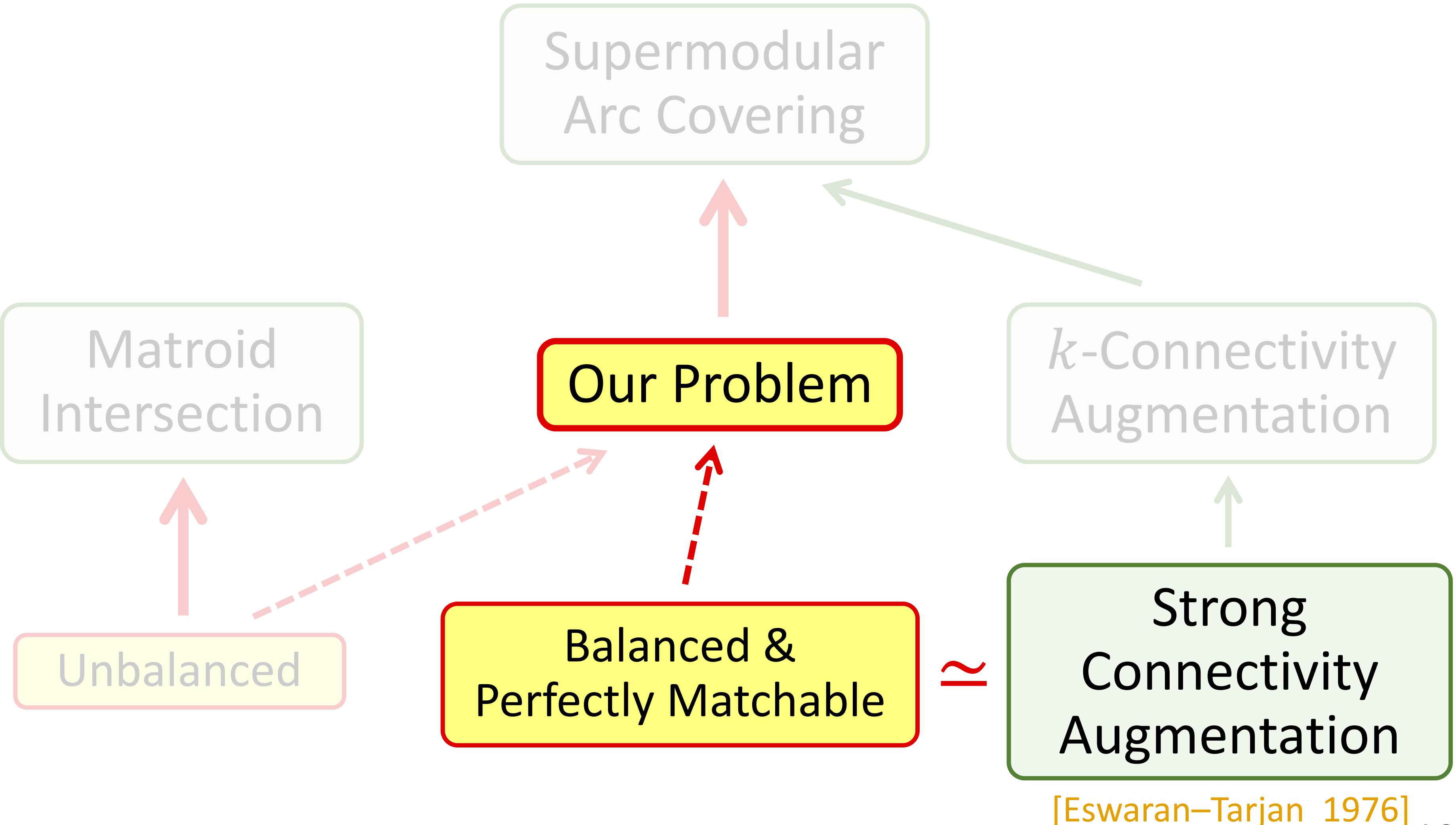
- **Min-Max Duality** via **Supermodular Arc Covering**
  [Frank–Jordán 1995]
- Unbalanced ($|V^+| \neq |V^-|$) $\subseteq$ **Matroid Intersection**
- Balanced ($|V^+| = |V^-|$) & Perfectly Matchable
  $\simeq$ **Strong Connectivity Augmentation** [Eswaran–Tarjan 1976]
- Balanced & NOT P.M. $\rightarrow$ **Direct** $O(nm)$-time **Algorithm**
  (Moreover, **General Case**)

Overview

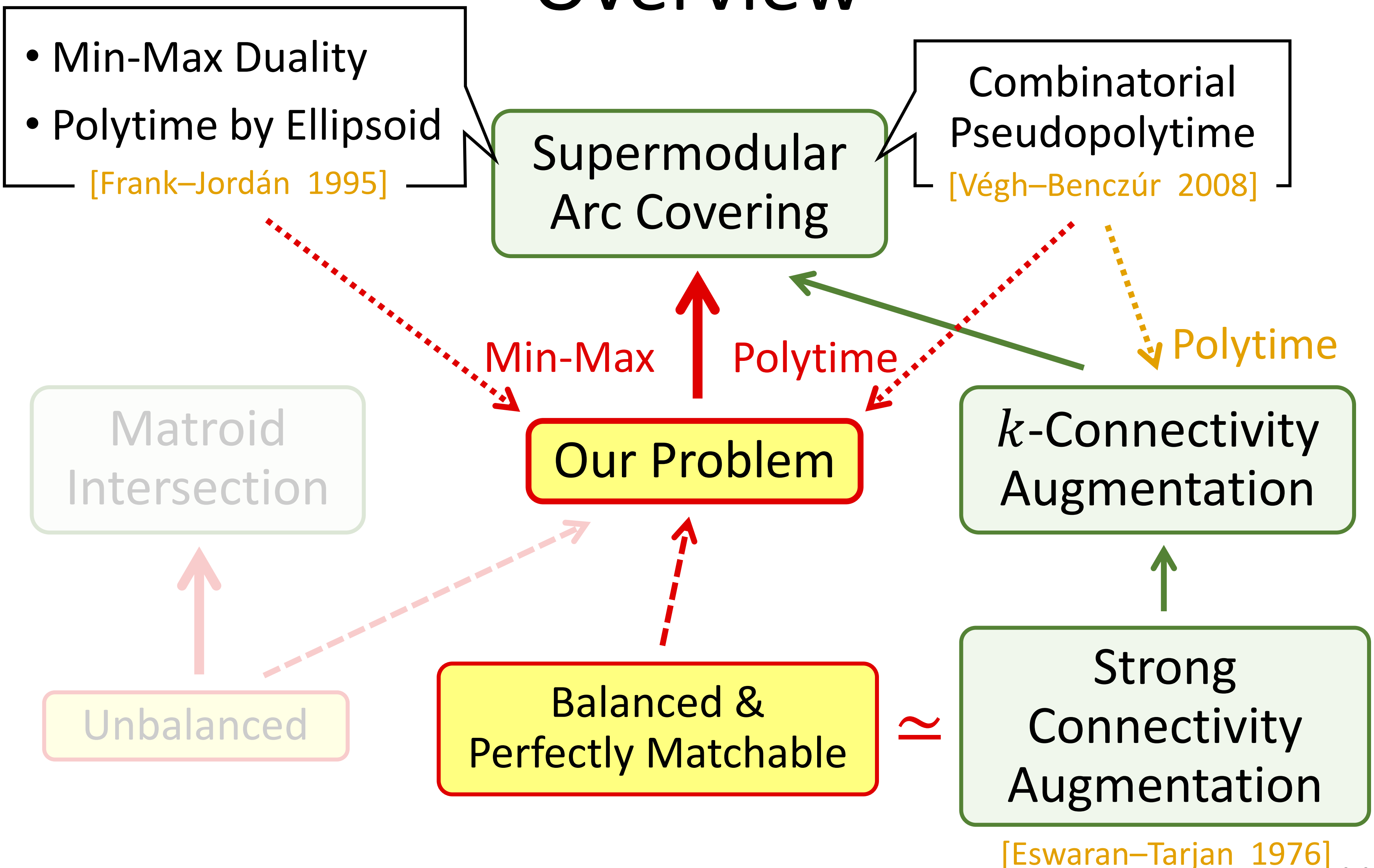


Supermodular Arc Covering
Matroid Intersection
Our Problem
k-Connectivity Augmentation
Unbalanced
Balanced & Perfectly Matchable
$\simeq$
Strong Connectivity Augmentation
[Eswaran–Tarjan 1976]

Overview

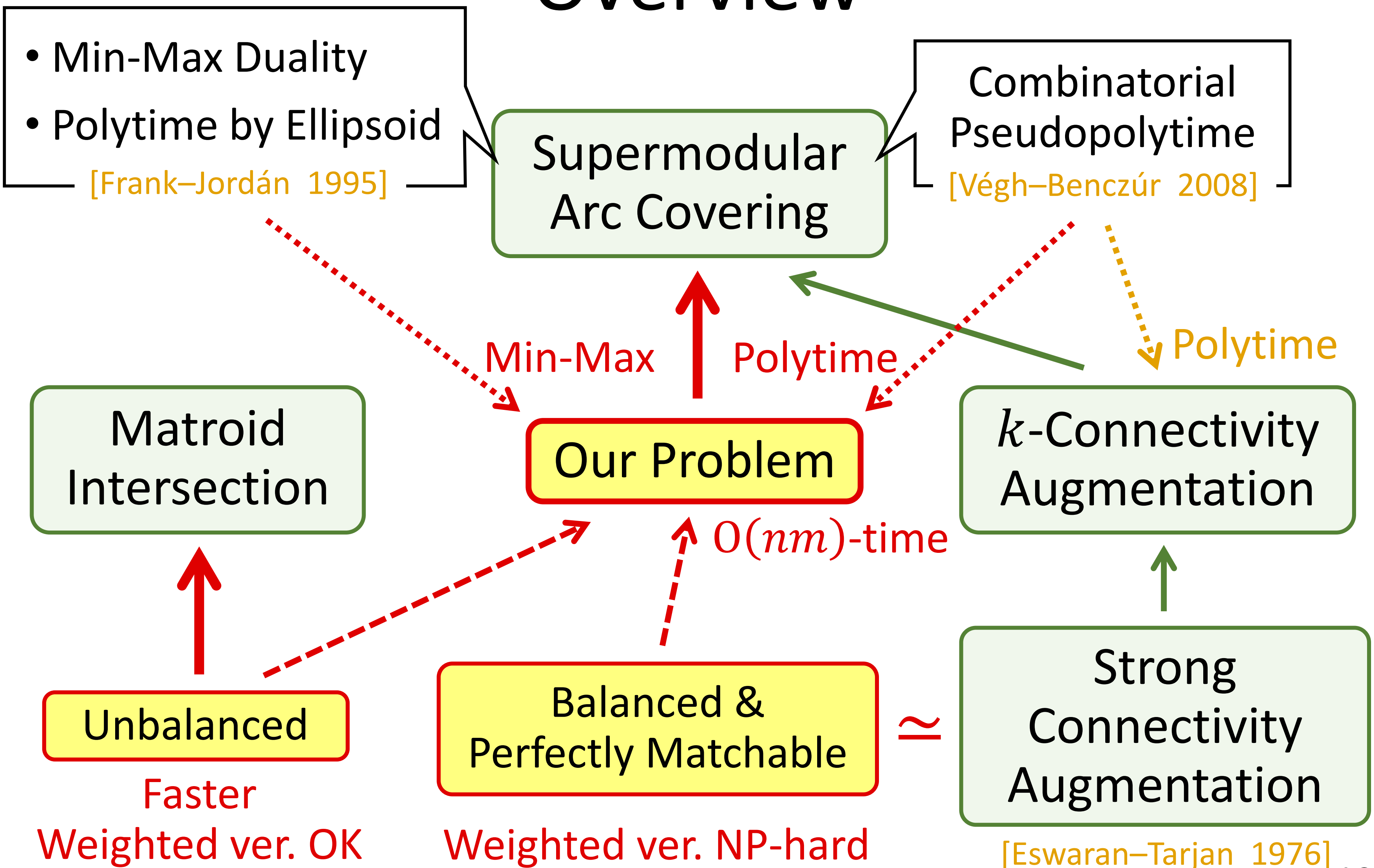

Supermodular
Arc Covering
Matroid
Intersection
Our Problem
k-Connectivity
Augmentation
Unbalanced
Balanced &
Perfectly Matchable
≃
Strong
Connectivity
Augmentation
[Eswaran–Tarjan 1976]

Overview



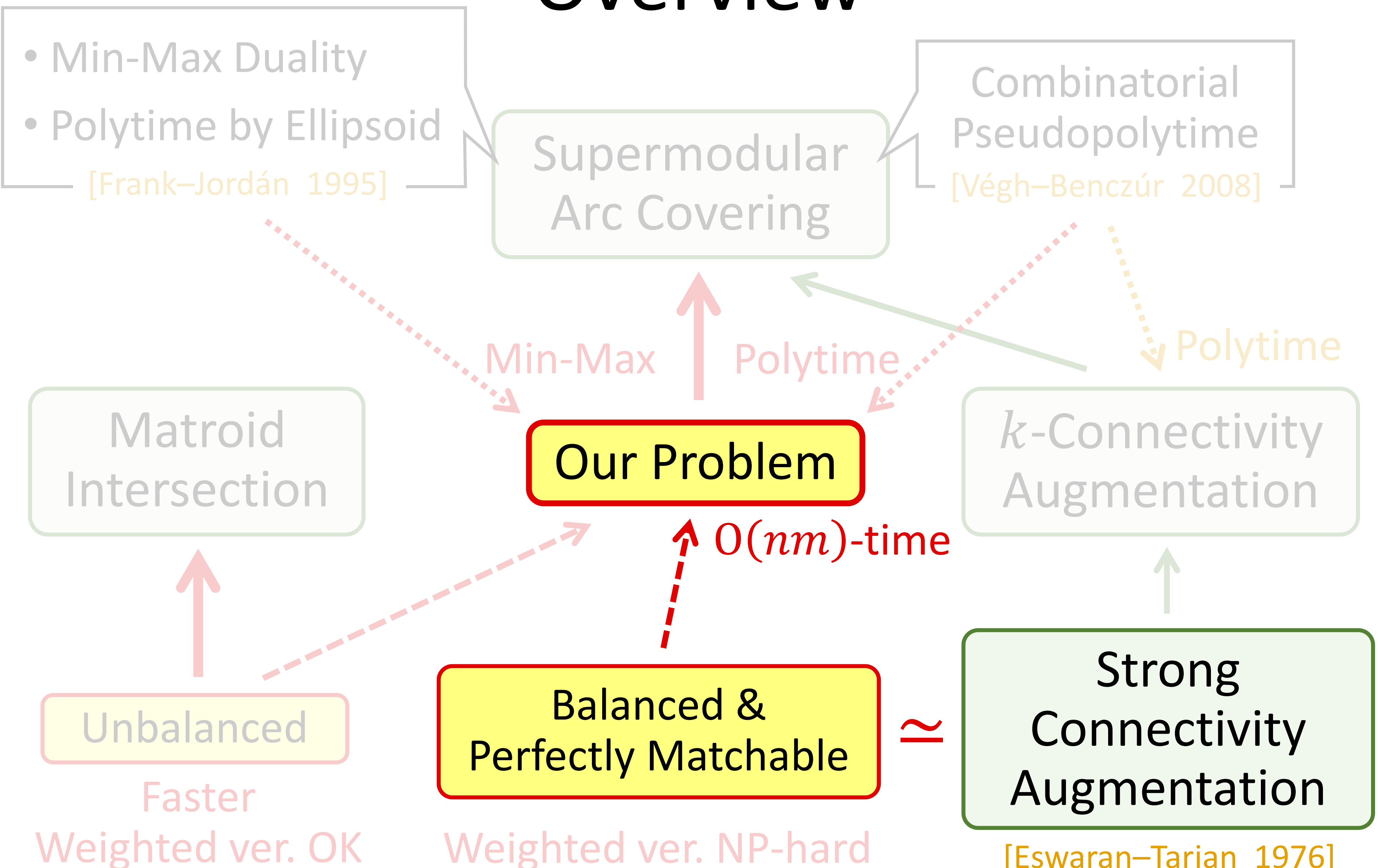
• Min-Max Duality
• Polytime by Ellipsoid
[Frank–Jordán 1995]
Supermodular Arc Covering
Combinatorial Pseudopolytime
[Végh–Benczúr 2008]
Min-Max
Polytime
Polytime
Matroid Intersection
Our Problem
k-Connectivity Augmentation
Unbalanced
Balanced & Perfectly Matchable
≃
Strong Connectivity Augmentation
[Eswaran–Tarjan 1976]

Overview



• Min-Max Duality
• Polytime by Ellipsoid
[Frank–Jordán 1995]
Supermodular Arc Covering
Combinatorial Pseudopolytime
[Végh–Benczúr 2008]
Min-Max
Polytime
Polytime
Matroid Intersection
Our Problem
k-Connectivity Augmentation
$O(nm)$-time
Unbalanced
Balanced & Perfectly Matchable
$\simeq$
Strong Connectivity Augmentation
Faster
Weighted ver. OK
Weighted ver. NP-hard
[Eswaran–Tarjan 1976]

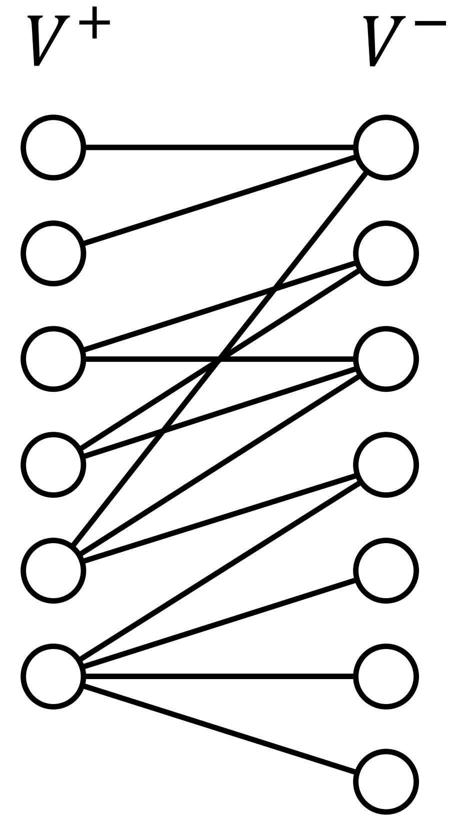
Overview



• Min-Max Duality
• Polytime by Ellipsoid
[Frank–Jordán 1995]
Supermodular Arc Covering
Combinatorial Pseudopolytime
[Végh–Benczúr 2008]
Min-Max
Polytime
Polytime
Matroid Intersection
Our Problem
k-Connectivity Augmentation
O(nm)-time
Unbalanced
Balanced & Perfectly Matchable
≃
Strong Connectivity Augmentation
Faster
Weighted ver. OK
Weighted ver. NP-hard
[Eswaran–Tarjan 1976]

How to Compute DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

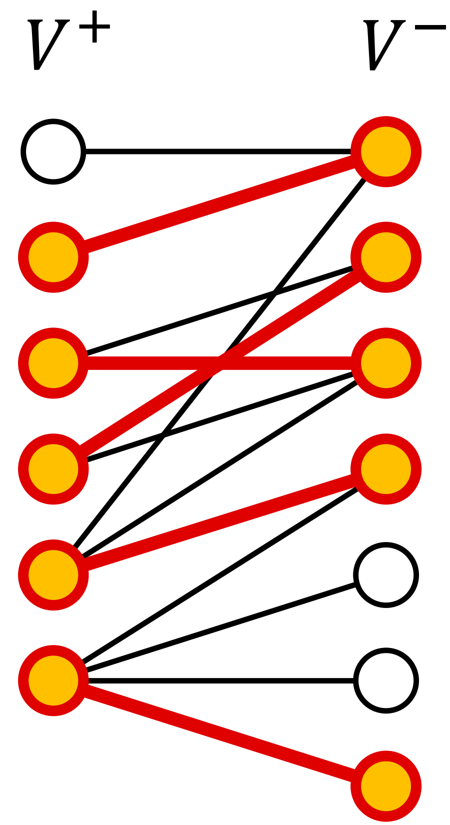


- Find a **Maximum Matching** M in G
- Orient Edges so that
  $M \implies$ **Both Directions** $\leftrightarrow$
  $E \setminus M \implies$ **Left to Right** $\rightarrow$
- V_0: **Reachable from** $V^+ \setminus \partial^+ M$
- V_∞: **Reachable to** $V^- \setminus \partial^- M$
- V_i: **Strongly Connected Component** of $G - V_0 - V_\infty$

How to Compute DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

- Find a **Maximum Matching** M in G
- Orient Edges so that
  $M \implies$ **Both Directions** $\leftrightarrow$
  $E \setminus M \implies$ **Left to Right** $\rightarrow$
- V_0: **Reachable from** $V^+ \setminus \partial^+ M$
- V_∞: **Reachable to** $V^- \setminus \partial^- M$
- V_i: **Strongly Connected Component** of $G - V_0 - V_\infty$



How to Compute DM-decomposition

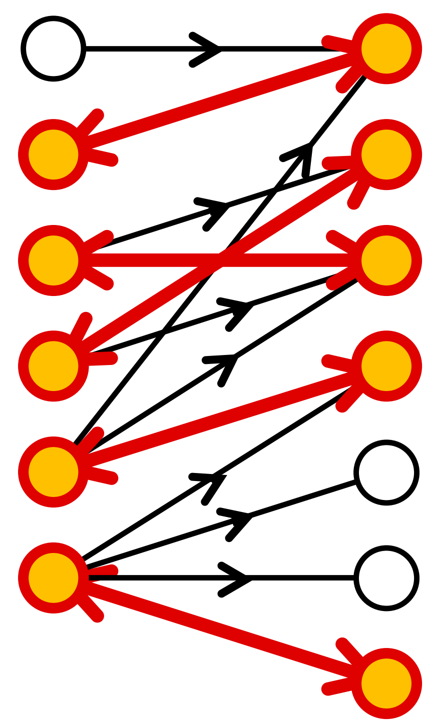
$G = (V^+, V^-; E)$: Bipartite Graph

- Find a **Maximum Matching** M in G
- Orient Edges so that

  $M \implies$ **Both Directions** $\leftrightarrow$

  $E \setminus M \implies$ **Left to Right** $\rightarrow$

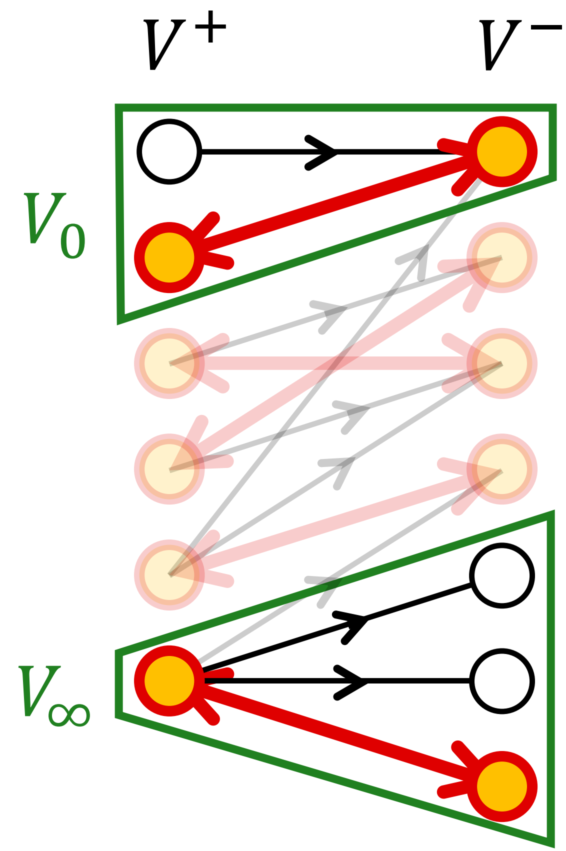
- V_0: **Reachable from** $V^+ \setminus \partial^+ M$
- V_∞: **Reachable to** $V^- \setminus \partial^- M$
- V_i: **Strongly Connected Component** of $G - V_0 - V_\infty$



How to Compute DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

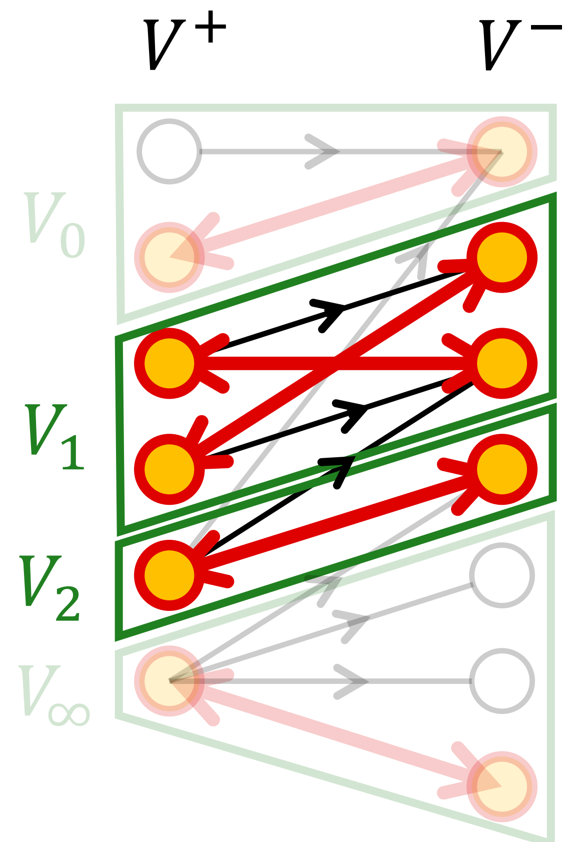
- Find a **Maximum Matching** M in G
- Orient Edges so that
  $M \implies$ **Both Directions** $\leftrightarrow$
  $E \setminus M \implies$ **Left to Right** $\rightarrow$
- V_0: Reachable from $V^+ \setminus \partial^+ M$
- V_∞: Reachable to $V^- \setminus \partial^- M$
- V_i: **Strongly Connected Component** of $G - V_0 - V_\infty$



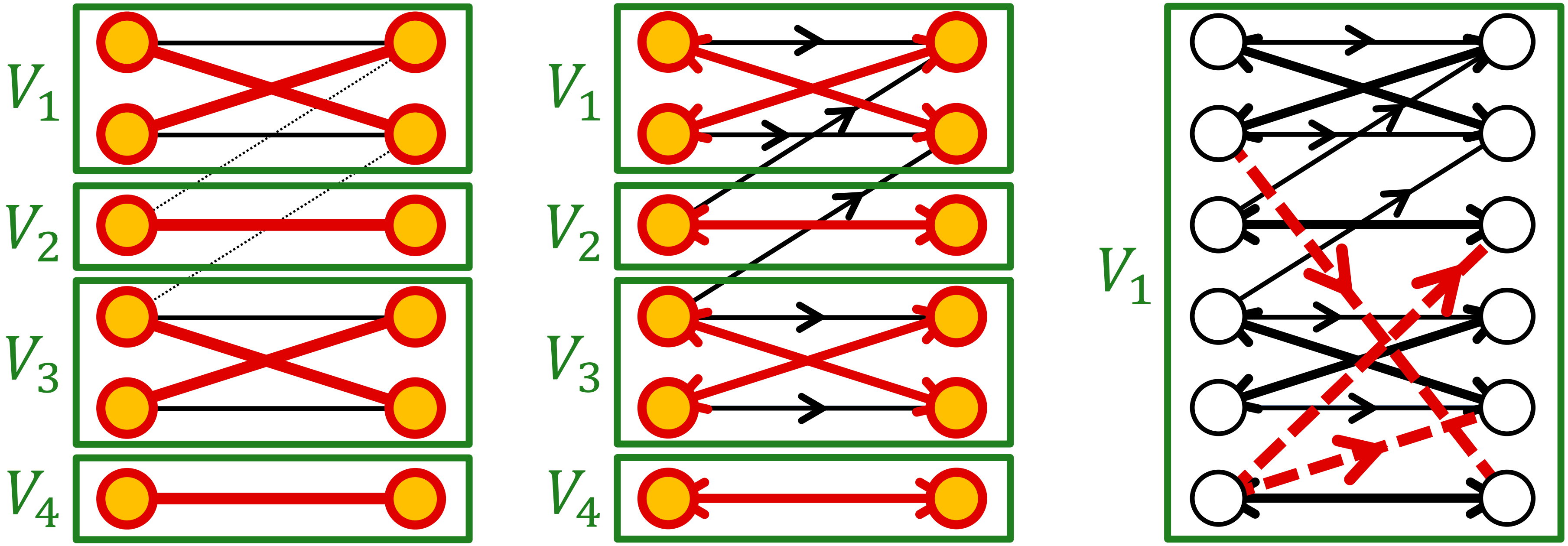
How to Compute DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

- Find a **Maximum Matching** M in G
- Orient Edges so that
  $M \implies$ **Both Directions** $\leftrightarrow$
  $E \setminus M \implies$ **Left to Right** $\rightarrow$
- V_0: **Reachable from** $V^+ \setminus \partial^+ M$
- V_∞: **Reachable to** $V^- \setminus \partial^- M$
- V_i: **Strongly Connected Component** of $G - V_0 - V_\infty$



When Balanced & Perfectly Matchable



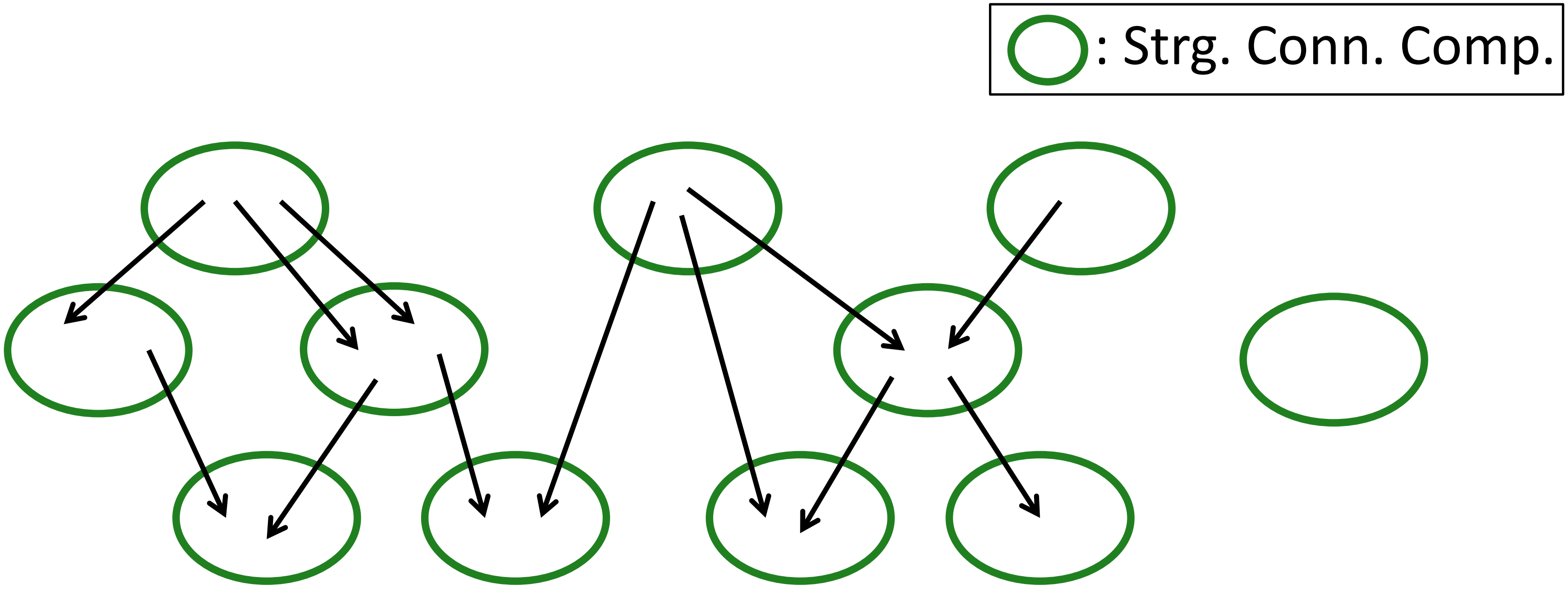
DM-decomposition $=$ **Decompoisition into Strg. Conn. Comps.** $\rightarrow$ **Make it Strg. Conn. by Adding Edges**

Obs. **DM-irreducibility** is **Equivalent to** **Strong Connectivity** of the Oriented Graph

Strong Connectivity Augmentation

Input $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

Goal Find a **Minimum Number of Additional Edges** to Make G **Strongly Connected**

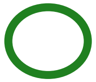


Strong Connectivity Augmentation

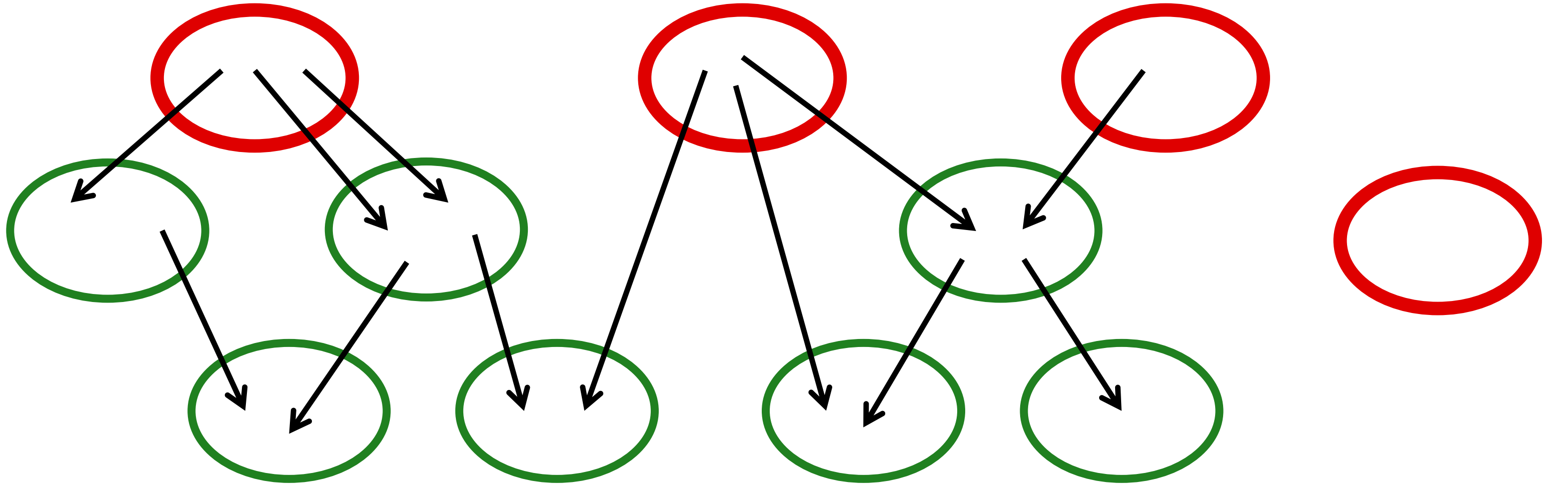
<u>Input</u> $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

<u>Goal</u> Find a **Minimum Number of Additional Edges** to Make G **<u>Strongly Connected</u>**

Each **<u>Source</u>** needs an **<u>Entering Edge</u>**



: Strg. Conn. Comp.

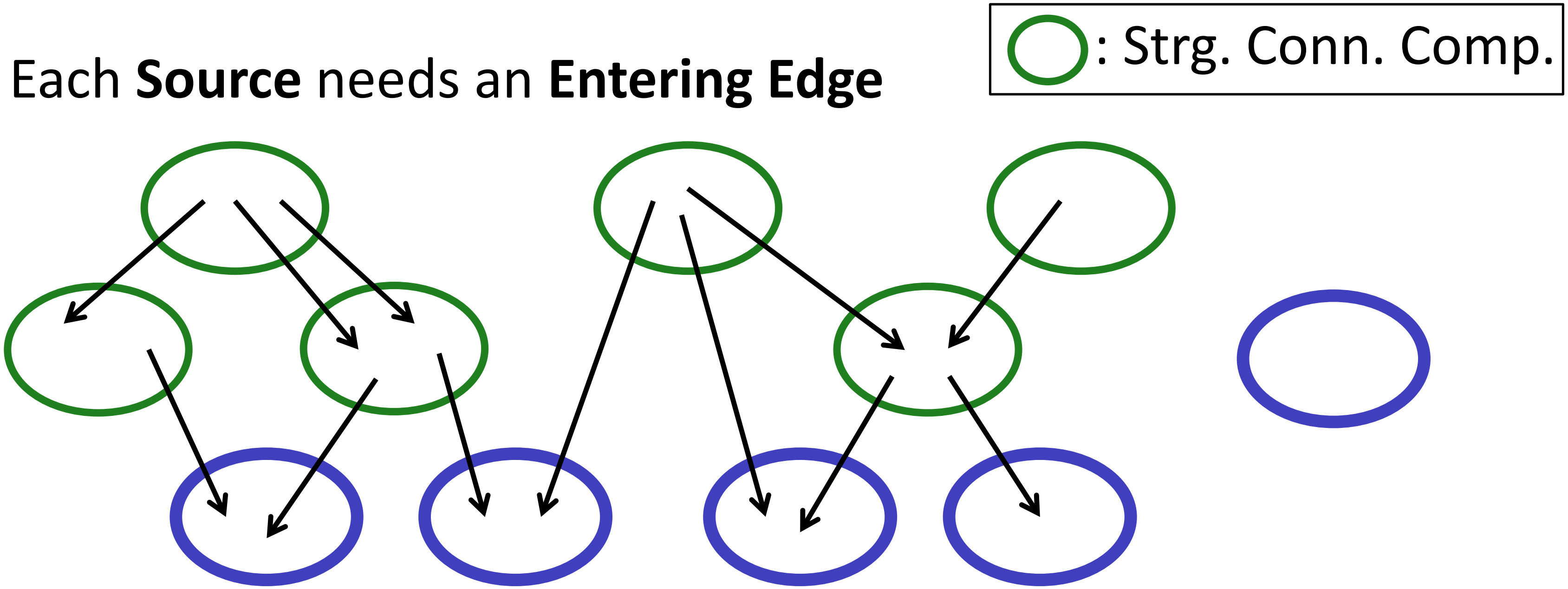


Strong Connectivity Augmentation

Input $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

Goal Find a **Minimum Number of Additional Edges** to Make G **Strongly Connected**

Each **Source** needs an **Entering Edge**



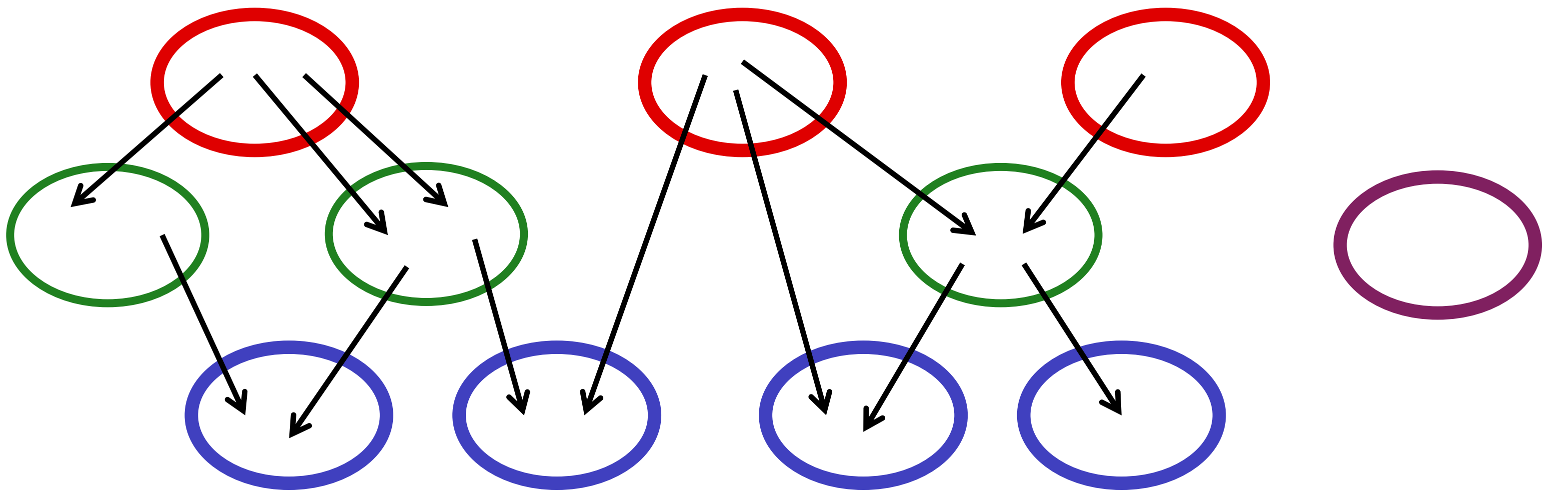
Each **Sink** needs a **Leaving Edge**

Strong Connectivity Augmentation

Input $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

Goal Find a **Minimum Number of Additional Edges** to Make G **Strongly Connected**

$\max\{$# of **Sources**, # of **Sinks**$\}$ edges are **Necessary**.



Strong Connectivity Augmentation

Input $G = (V, E)$: Directed Graph (NOT Strg. Conn.)

Goal Find a **Minimum Number of Additional Edges** to Make G **Strongly Connected**

$\max$ {# of **Sources**, # of **Sinks**} edges are **Necessary**.

Thm. It is also **Sufficient**.

One can find such an edge set in **Linear Time**.

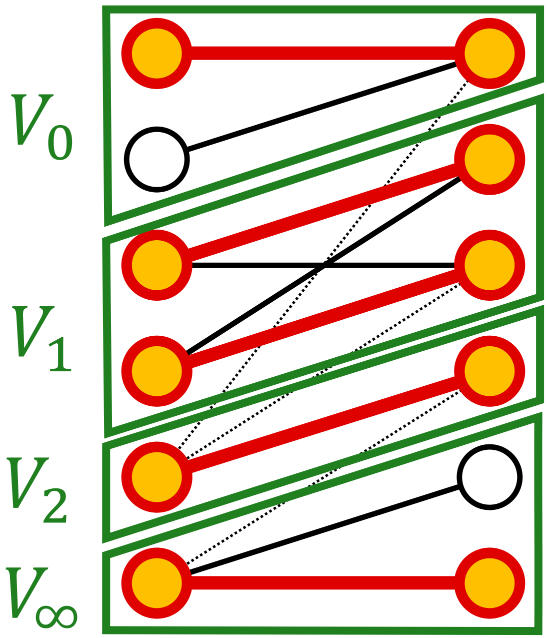
[Eswaran–Tarjan 1976]

Cor. If the input is **Balanced with Perfect Matching**, Our Problem can be solved in **Linear Time**.

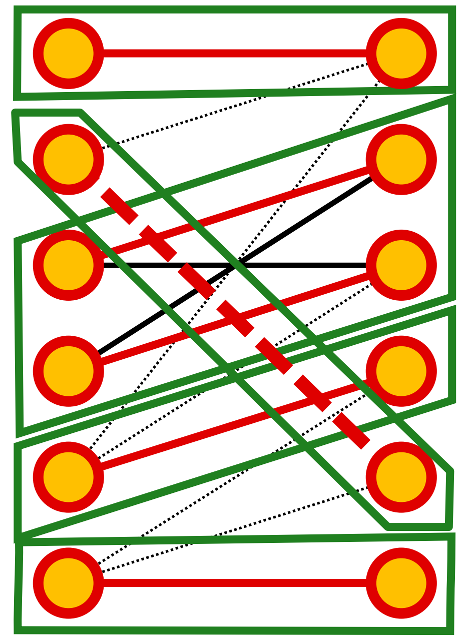
When Balanced & NOT Perfectly Matchable

<u>Idea</u> Reduce to P.M. Case by **Connecting** **Exposed Vertices**

Each V_i $(i \neq 0, \infty)$ remains as it was



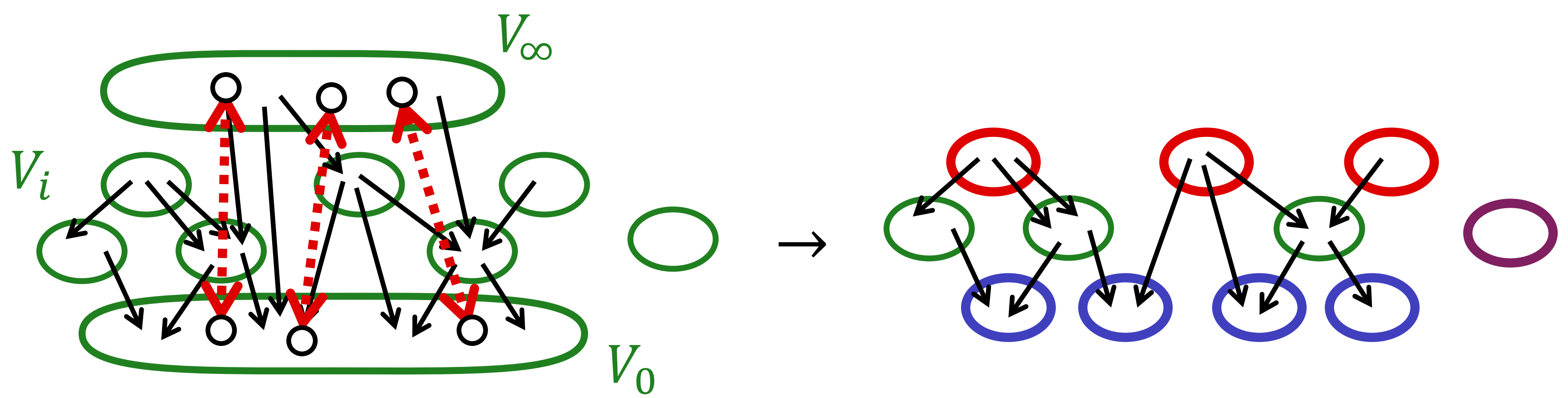
→



When Balanced & NOT Perfectly Matchable

Idea Reduce to P.M. Case by **Connecting** **Exposed Vertices**

Each V_i $(i \neq 0, \infty)$ remains as it was



$|V^+| - |M|$ + max {# of **Sources**, # of **Sinks**}

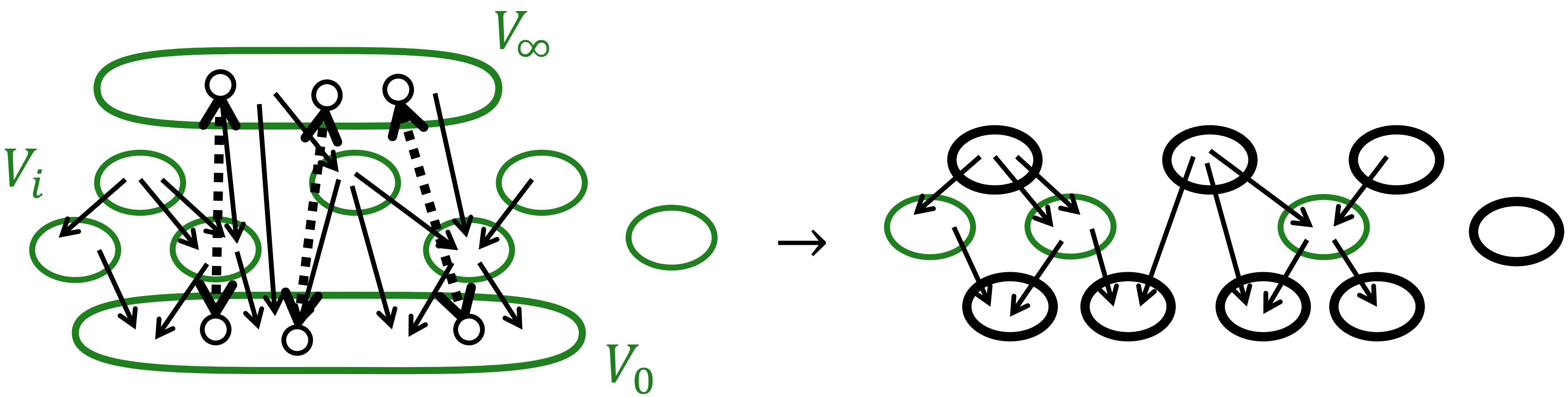
||

of **Additional Edges**

When Balanced & NOT Perfectly Matchable

<u>Idea</u> Reduce to P.M. Case by **Connecting Exposed Vertices**

Each V_i $(i \neq 0, \infty)$ remains as it was



$$\underbrace{|V^+| - |M|}_{\textbf{Const.}} + \max\{\underbrace{\text{\# of }\textbf{Sources}\text{, \# of }\textbf{Sinks}}_{\textbf{Depending on } M}\}$$

||

of **Additional Edges**

When Balanced & NOT Perfectly Matchable

Idea Reduce to P.M. Case by **Connecting Exposed Vertices**

- (# of **Sources** or of **Sinks**) depends on Max. Matching M
- Find **Eligible Perfect Matchings** in $G[V_\infty]$ and in $G[V_0]$
  - Minimizing (# of **Sources in** V_∞) and (# of **Sinks in** V_0)
  - Just by finding two edge-disjoint s–t paths $\mathrm{O}(n)$ times
- Optimality is guaranteed by **Min-Max Duality**

Thm. If the input is Balanced (in fact, NOT necessary),
Our Problem can be solved in $\mathrm{O}(nm)$ time.

[BIKY 2018]

When Balanced & NOT Perfectly Matchable

Idea Reduce to P.M. Case by **Connecting Exposed Vertices**

- (# of **Sources** or of **Sinks**) depends on Max. Matching M
- Find **Eligible Perfect Matchings** in $G[V_\infty]$ and in $G[V_0]$
  - Minimizing (# of **Sources in** V_∞) and (# of **Sinks in** V_0)
  - Just by finding two edge-disjoint s–t paths $O(n)$ times
- Optimality is guaranteed by **Min-Max Duality**

Thm. If the input is Balanced (in fact, NOT necessary), Our Problem can be solved in $O(nm)$ time.

[BIKY 2018]

When Balanced & NOT Perfectly Matchable

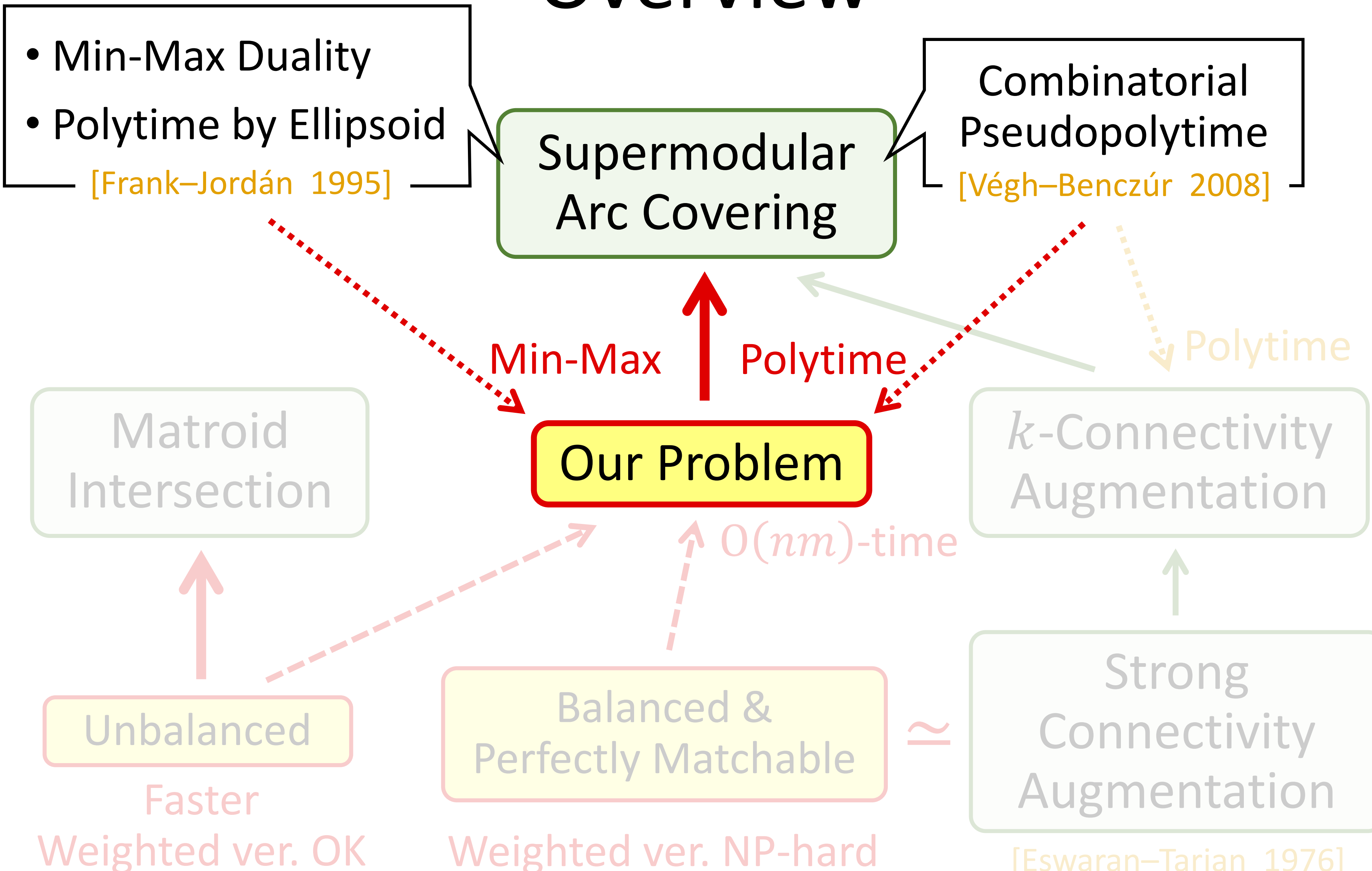
Idea Reduce to P.M. Case by **Connecting Exposed Vertices**

- (# of **Sources** or of **Sinks**) depends on Max. Matching M
- Find **Eligible Perfect Matchings** in $G[V_\infty]$ and in $G[V_0]$
  - Minimizing (# of **Sources in** V_∞) and (# of **Sinks in** V_0)
  - Just by finding two edge-disjoint s–t paths $\mathrm{O}(n)$ times
- Optimality is guaranteed by **Min-Max Duality**

Thm. If the input is Balanced (in fact, NOT necessary), Our Problem can be solved in $\mathrm{O}(nm)$ time.

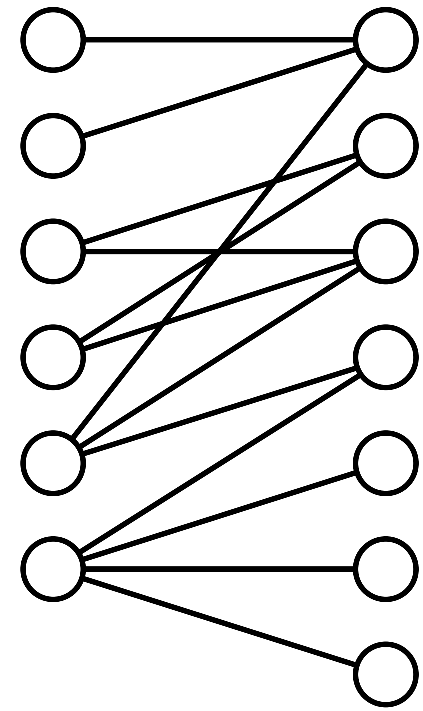
[BIKY 2018]

Overview


• Min-Max Duality
• Polytime by Ellipsoid
[Frank–Jordán 1995]
Supermodular Arc Covering
Combinatorial Pseudopolytime
[Végh–Benczúr 2008]
Min-Max
Polytime
Polytime
Matroid Intersection
Our Problem
k-Connectivity Augmentation
O(nm)-time
Unbalanced
Faster
Weighted ver. OK
Balanced & Perfectly Matchable
Weighted ver. NP-hard
≃
Strong Connectivity Augmentation
[Eswaran–Tarjan 1976]

Definition of DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph



$f_G(X^+) := |\Gamma_G(X^+)| - |X^+|$ $(X^+ \subseteq V^+)$

(Surplus for Hall's Condition)

f_G is Submodular

- Minimizers form Distributive Lattice
- $X_0^+ \subsetneq X_1^+ \subsetneq \cdots \subsetneq X_k^+$: Maximal Chain

$V_0^+ := X_0^+,\qquad V_0^- := \Gamma_G(X_0^+)$

$V_i^+ := X_i^+ \setminus X_{i-1}^+,\qquad V_i^- := \Gamma_G(X_i^+) \setminus \Gamma_G(X_{i-1}^+)$

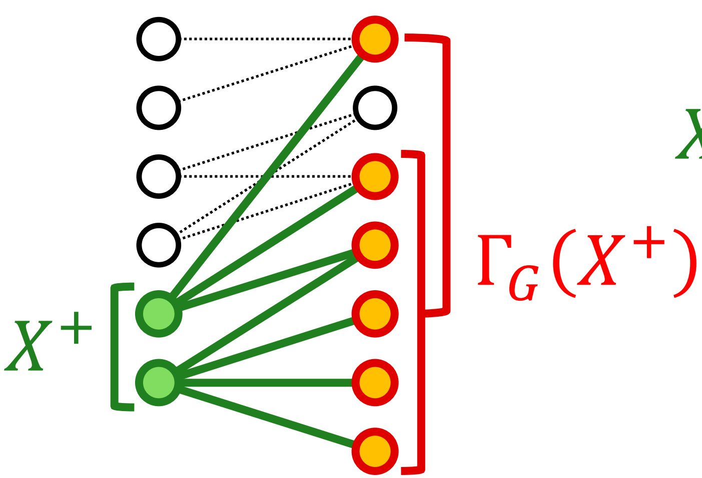
$V_\infty^+ := V^+ \setminus X_k^+,\qquad V_\infty^- := V^- \setminus \Gamma_G(X_k^+)$

Definition of DM-decomposition

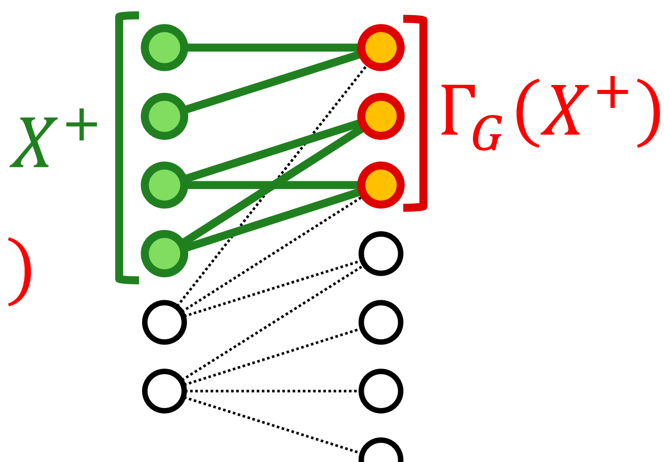
$G = (V^+, V^-; E)$: Bipartite Graph

$f_G(X^+) \coloneqq |\Gamma_G(X^+)| - |X^+| \ (X^+ \subseteq V^+)$

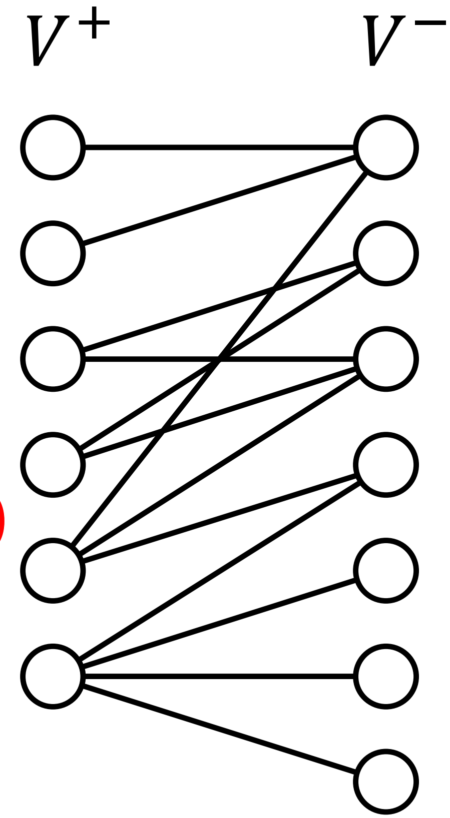
(Surplus for Hall's Condition)



$f_G = 4$

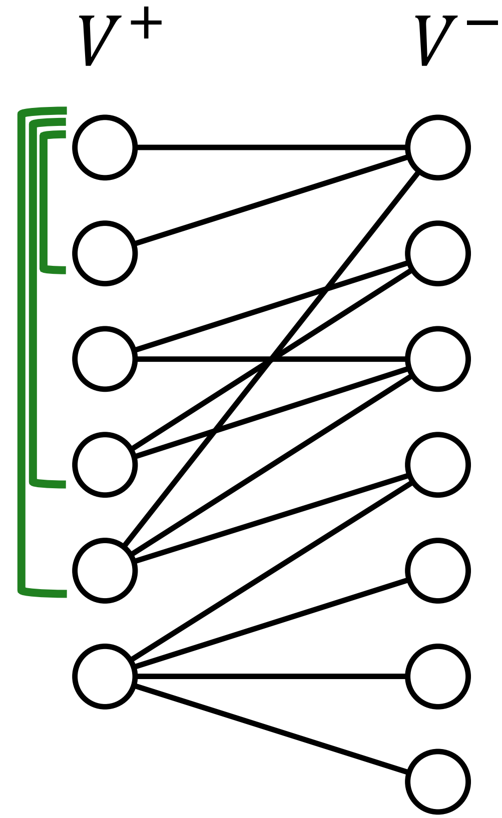


$f_G = -1$



Definition of DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

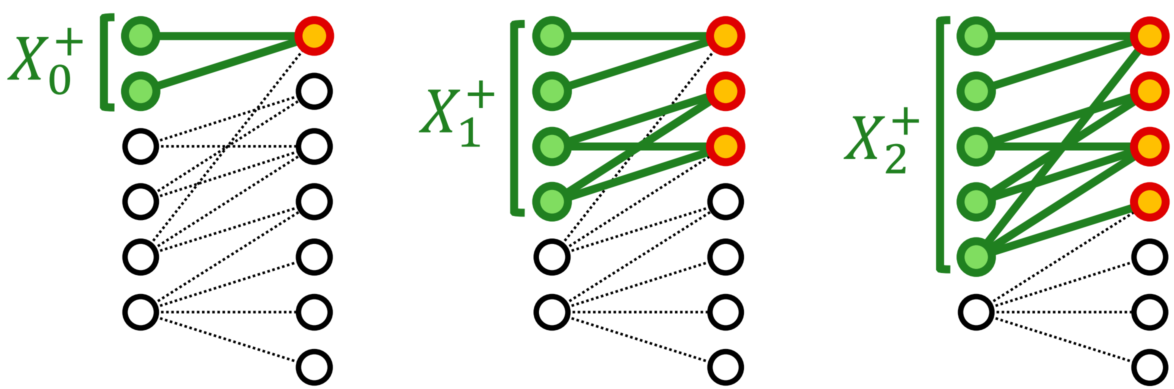


$f_G(X^+) := |\Gamma_G(X^+)| - |X^+| \ \ (X^+ \subseteq V^+)$

(Surplus for Hall's Condition)

f_G is Submodular

- Minimizers form Distributive Lattice
- $X_0^+ \subsetneq X_1^+ \subsetneq \cdots \subsetneq X_k^+$: Maximal Chain

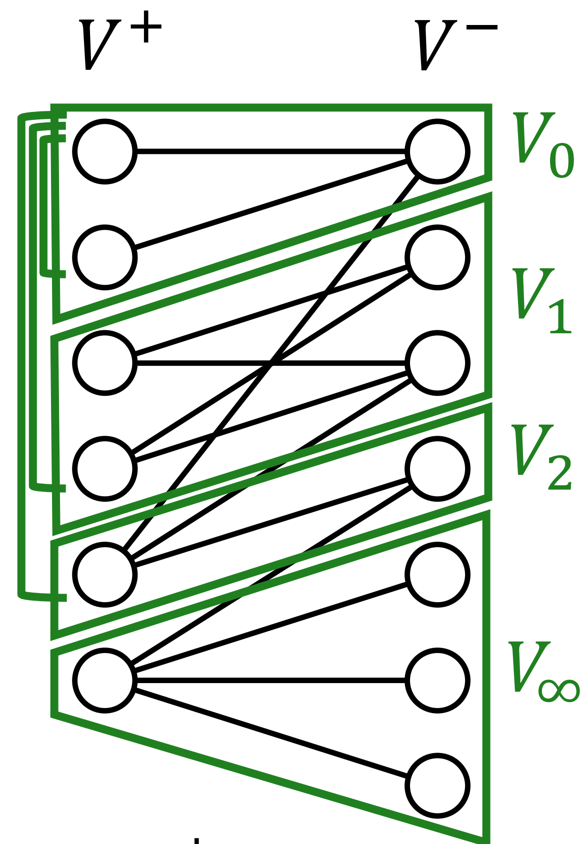


Definition of DM-decomposition

$G = (V^+, V^-; E)$: Bipartite Graph

$f_G(X^+) := |\Gamma_G(X^+)| - |X^+| \;\; (X^+ \subseteq V^+)$

(Surplus for Hall's Condition)



f_G is Submodular

- Minimizers form Distributive Lattice
- $X_0^+ \subsetneq X_1^+ \subsetneq \cdots \subsetneq X_k^+$: Maximal Chain

$$V_0^+ := X_0^+, \qquad V_0^- := \Gamma_G(X_0^+)$$

$$V_i^+ := X_i^+ \setminus X_{i-1}^+, \qquad V_i^- := \Gamma_G(X_i^+) \setminus \Gamma_G(X_{i-1}^+)$$

$$V_\infty^+ := V^+ \setminus X_k^+, \qquad V_\infty^- := V^- \setminus \Gamma_G(X_k^+)$$

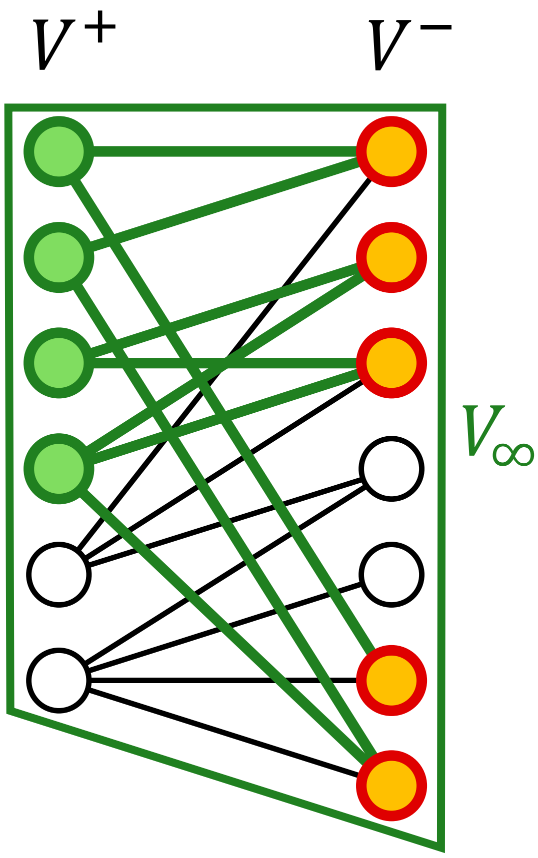
Rephrasing of DM-irreducibility

$G = (V^+, V^-; E)$: Bipartite Graph

$f_G(X^+) := |\Gamma_G(X^+)| - |X^+|$ $(X^+ \subseteq V^+)$

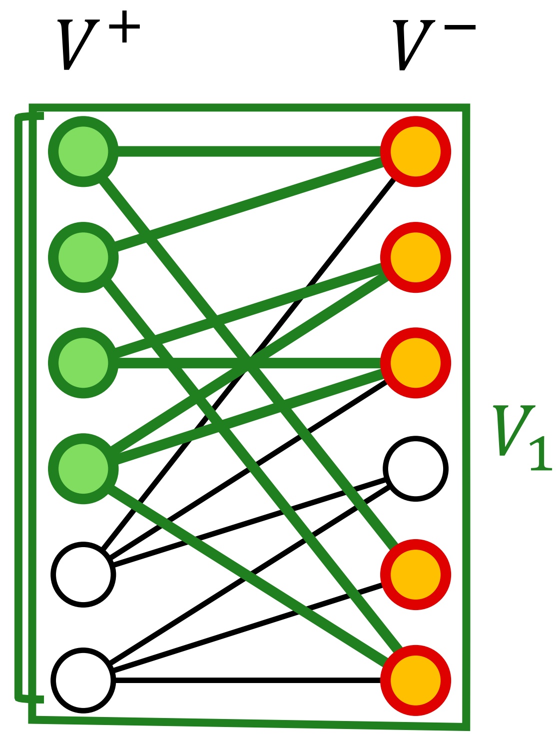
(Surplus for Hall's Condition)

- When $|V^+| < |V^-|$ (Unbalanced)
  $\rightarrow$ $\emptyset \subseteq V^+$ is a unique minimizer
  $\Leftrightarrow$ $|\Gamma_G(X^+)| > |X^+|$ $(\emptyset \neq \forall X^+ \subseteq V^+)$
- When $|V^+| = |V^-|$ (Balanced)
  $\rightarrow$ Only $\emptyset$ and V^+ are minimizers
  $\Leftrightarrow$ $|\Gamma_G(X^+)| > |X^+|$ $(\emptyset \neq \forall X^+ \subsetneq V^+)$



Rephrasing of DM-irreducibility

$G = (V^+, V^-; E)$: Bipartite Graph



$f_G(X^+) \coloneqq |\Gamma_G(X^+)| - |X^+| \ \ (X^+ \subseteq V^+)$

(Surplus for Hall's Condition)

- When $|V^+| < |V^-|$ (Unbalanced)

  $\rightarrow$ $\emptyset \subseteq V^+$ is a unique minimizer

  $\Leftrightarrow$ $|\Gamma_G(X^+)| > |X^+| \ \ (\emptyset \neq \forall X^+ \subseteq V^+)$

- When $|V^+| = |V^-|$ (Balanced)

  $\rightarrow$ Only $\emptyset$ and V^+ are minimizers

  $\Leftrightarrow$ $|\Gamma_G(X^+)| > |X^+| \ \ (\emptyset \neq \forall X^+ \subsetneq V^+)$

Symmetrically for V^-

Min-Max Duality (Unbalanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)

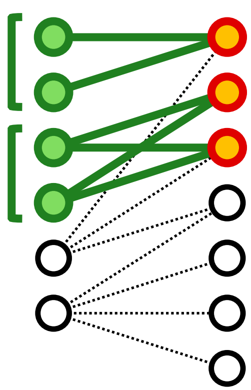
Goal Find a **Smallest Set** F **of Additional Edges** s.t. $|\Gamma_{G+F}(X^+)| > |X^+|$ $(\emptyset \neq \forall X^+ \subseteq V^+)$

$$|F| \geq \sum_{X^+ \in \mathcal{X}^+} \left(1 - f_G(X^+)\right) \quad (\forall \mathcal{X}^+\text{: Subpartition of } V^+)$$

$f_G(X^+) = |\Gamma_G(X^+)| - |X^+|$

$f_G = -1$

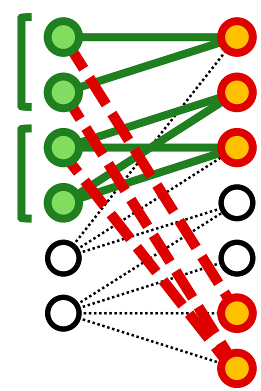
$f_G = 0$



$\rightarrow$

$f_{G+F} = 1$

$f_{G+F} = 1$



Min-Max Duality (Unbalanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)

Goal Find a **Smallest Set F of Additional Edges**
s.t. $|\Gamma_{G+F}(X^+)| > |X^+|$ $(\emptyset \neq \forall X^+ \subseteq V^+)$

$$|F| \geq \sum_{X^+ \in \mathcal{X}^+} \left(1 - f_G(X^+)\right) \quad (\forall \mathcal{X}^+\text{: Subpartition of } V^+)$$

$f_G(X^+) = |\Gamma_G(X^+)| - |X^+|$

Thm. $\min\{\, |F| \mid G + F \text{ is DM–irreducible} \,\}$
$=$
$\max\left\{ \sum_{X^+ \in \mathcal{X}^+} \left(1 - f_G(X^+)\right) \mid \mathcal{X}^+\text{: Subpartition of } V^+ \right\}$

[BIKY 2018]

Min-Max Duality (Balanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| = |V^-|$)

Goal Find a **Smallest Set** F **of Additional Edges**
s.t. $\left|\Gamma_{G+F}(X^\pm)\right| > \left|X^\pm\right|$ $(\emptyset \neq \forall X^\pm \subsetneq V^\pm)$

Thm. $\min\{\,|F| \mid G + F$ is DM–irreducible $\}$

$\|$ $\mathcal{X}^\pm \neq \{V^\pm\}$

$$\max\left\{\begin{aligned}&\max\{\,\tau_G(\mathcal{X}^+) \mid \mathcal{X}^+\text{: \textbf{Proper} Subpartition of } V^+\,\},\\&\max\{\,\tau_{\overline{G}}(\mathcal{X}^-) \mid \mathcal{X}^-\text{: \textbf{Proper} Subpartition of } V^-\,\}\end{aligned}\right\}$$

[BIKY 2018]

$$\tau_G(\mathcal{X}^+) \coloneqq \sum_{X^+\in\mathcal{X}^+}\bigl(1 - f_G(X^+)\bigr)$$

$\overline{G} \coloneqq \left(V^-, V^+; \overline{E}\right)$: Interchanging V^+ and V^-

Supermodular Arc Covering

Thm. V^+, V^-: Finite Sets (possibly intersecting)
$\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$: Crossing Family (Constraint Set)
$g: \mathcal{F} \to \mathbf{Z}_{\geq 0}$ Supermodular (Demand on $\mathcal{F}$)

The minimum cardinality of a multiset $A: V^+ \times V^- \to \mathbf{Z}_{\geq 0}$ of directed edges in $V^+ \times V^-$ that **covers** g is equal to

$\max_{\mathcal{S} \subseteq \mathcal{F}} \left\{ \sum_{(X^+, X^-) \in \mathcal{S}} g(X^+, X^-) \;\middle|\; \mathcal{S}\text{: } \textbf{pairwise independent} \right\}$

[Frank–Jordán 1995]

- Packing (Max) vs. Covering (Min) type Strong Duality
- Polytime Solvability by Ellipsoid Method
- Including Directed k-Connectivity Augmentation etc.

Supermodular Arc Covering

Thm. V^+, V^-: Finite Sets (possibly intersecting)
$\mathcal{F} \subseteq 2^{V^+} \times 2^{V^-}$: Crossing Family (Constraint Set)
$g: \mathcal{F} \to \mathbf{Z}_{\geq 0}$ Supermodular (Demand on $\mathcal{F}$)

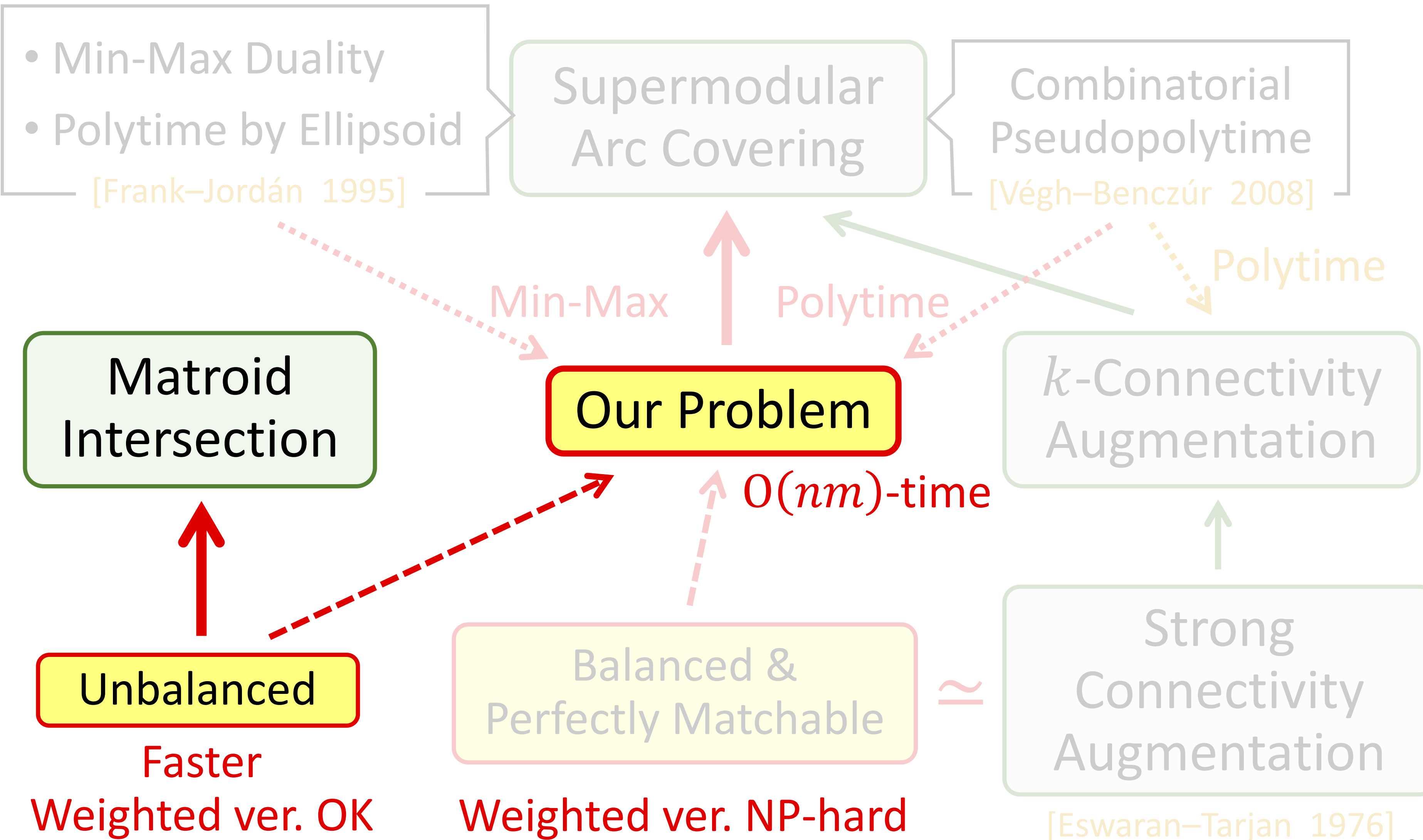
The minimum cardinality of a multiset $A: V^+ \times V^- \to \mathbf{Z}_{\geq 0}$ of directed edges in $V^+ \times V^-$ that **covers** g is equal to

$$\max_{\mathcal{S} \subseteq \mathcal{F}} \left\{ \sum_{(X^+, X^-) \in \mathcal{S}} g(X^+, X^-) \;\middle|\; \mathcal{S}: \textbf{pairwise independent} \right\}$$

[Frank–Jordán 1995]

By defining $\mathcal{F}$ and g appropriately for Our Problem, we obtain Min-Max Duality Theorems for Our Problem (via some nontrivial Uncrossing arguments)

Overview


• Min-Max Duality
• Polytime by Ellipsoid
[Frank–Jordán 1995]
Supermodular Arc Covering
Combinatorial Pseudopolytime
[Végh–Benczúr 2008]
Polytime
Min-Max
Polytime
Matroid Intersection
Our Problem
k-Connectivity Augmentation
O(nm)-time
Unbalanced
Balanced & Perfectly Matchable
≃
Strong Connectivity Augmentation
[Eswaran–Tarjan 1976]
Faster
Weighted ver. OK
Weighted ver. NP-hard

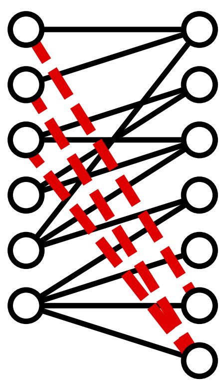
Weighted Problem (Unbalanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)
$c: (V^+ \times V^-) \setminus E \to \mathbf{R}_{>0}$ (Cost on Addition)

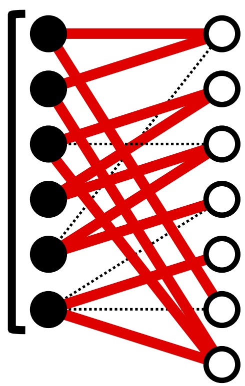
Goal Find a **Minimum-Cost Set** F **of Additional Edges** s.t. $G + F$ is **DM-irreducible**

Lem. $\exists G' \subseteq G + F$: **Forest** s.t. $|\Gamma_{G'}(v)| = 2$ $(\forall v \in V^+)$

[BIKY 2018]



$|\Gamma_{G'}(v)| = 2$



Minimally DM-irreducible

Weighted Problem (Unbalanced Case)

Input $G = (V^+, V^-; E)$: Bipartite Graph ($|V^+| < |V^-|$)

$c: (V^+ \times V^-) \setminus E \to \mathbf{R}_{>0}$ (Cost on Addition)

Goal Find a **Minimum-Cost Set F of Additional Edges** s.t. $G + F$ is **DM-irreducible**

Lem. $\exists G' \subseteq G + F$: **Forest** s.t. $|\Gamma_{G'}(v)| = 2$ ($\forall v \in V^+$)

[BIKY 2018]

Reduce to find a **Min-Weight Common Base** in

$\mathbf{M}_1$: Graphic Matroid (with $2|V^+|$-Truncation)

$\mathbf{M}_2$: Partition Matroid (Degree Constraint on V^+)

$$\gamma: V^+ \times V^- \to \mathbf{R}_{\geq 0} \text{ (Weight)};\ \gamma(e) := \begin{cases} 0 & (e \in E) \\ c(e) & (e \notin E) \end{cases}$$

Our Results (Summary)

Input $G = (V^+, V^-; E)$: Bipartite Graph

Goal Find a **Minimum Number of Additional Edges** to Make G **DM-irreducible**

- **Min-Max Duality** via **Supermodular Arc Covering**
  [Frank–Jordán 1995]
- Unbalanced ($|V^+| \neq |V^-|$) $\subseteq$ **Matroid Intersection**
- Balanced ($|V^+| = |V^-|$) & Perfectly Matchable
  $\simeq$ **Strong Connectivity Augmentation** [Eswaran–Tarjan 1976]
- Balanced & NOT P.M. $\rightarrow$ **Direct** $O(nm)$-time **Algorithm**

(Moreover, **General Case**)