Matroid Intersection under Restricted Oracles

Yutaro Yamaguchi

Collaborators: Kristóf Bérczi, Tamás Király, Yu Yokoi

Special Thanks: Mihály Bárász, Yuni Iwamasa, Taihei Oki

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Matroid Intersection under Restricted Oracles

Q. Is “Matroid Intersection” tractable? In what sense?

K. Bérczi, T. Király, Y. Yamaguchi, Y. Yokoi:
Matroid Intersection under Restricted Oracles.
SIAM Journal on Discrete Mathematics (SIDMA). To appear. (arXiv:2209.14516)

M. Bárász, K. Bérczi, T. Király, Y. Yamaguchi, Y. Yokoi:
Matroid Intersection under Minimum Rank Oracle. In preparation.
(Including and Extending
M. Bárász: **Matroid Intersection for the Min-Rank Oracle**.
EGRES Technical Report, QP-2006-03, 2006.)

To be continued…

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To be continued… and there are also spin-off papers:

K. Bérczi, T. Király, Y. Yamaguchi, Y. Yokoi:
Approximation by Lexicographically Maximal Solutions in Matching and Matroid Intersection Problems.
Theoretical Computer Science, **910** (2022), pp. 48–53.

K. Bérczi, T. Király, T. Schwarcz, Y. Yamaguchi, Y. Yokoi: **Hypergraph Characterization of Split Matroids**.
Journal of Combinatorial Theory, Series A, **194** (2023), No. 105697.

Outline

- Overview: Question and Results
- Matroid Intersection (Basics)
  - Matroid and Matroid Intersection
  - Augmenting-Path Algorithms and Exchangeability Graph
- Matroid Intersection under Restricted Oracles
  - First Step: What can be done in general by Common Independence Oracle
  - Results on Each Restricted Oracle
- Matroid Intersection under Minimum Rank Oracle
  - How to Solve Unweighted Problem
  - Results on Weighted Problem

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What is Matroid Intersection?

The intersection of two matroids

- Efficient Algorithms and Max-Min Theorems
  - A maximum-cardinality common independent set
  - A maximum-weight common independent set (of each cardinality)
- LP Formulation
  - Intersection of matroid polytopes = matroid intersection polytope
  - Total dual integrality (TDI) and well-structured dual solution
- Many Applications (= Unified Framework)

  Bipartite matching, Arborescence (packing), Dijoin, etc.

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Most of them require separate information

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What is known

- Efficient Algorithms rely on **separate** oracles for the two matroids
- Max-Min Theorems and the Polyhedral Description are given by using the two rank functions **separately**

What may be asked

- The resulting combinatorial structure is just $\mathcal{I}_1 \cap \mathcal{I}_2$
- The polytope is completely determined by $r_{\min} = \min\{r_1, r_2\}$
- When it is seen as a special case of Matroid Matching, the input should be $r_{\text{sum}} = r_1 + r_2$ (oracle)

"Matroid Intersection is **Tractable**"

Most of them require separate information of two matroids

Basic Strategy of Efficient Algorithms

- Starting with $Y = \emptyset$, repeatedly update the current solution Y.
- For each update,
  - construct the exchangeability graph w.r.t. Y,
  - find an augmenting path P in the graph, and
  - flip the current solution along the path, i.e., $Y \leftarrow Y \triangle P$.
- The edges in the graph are oriented according to **in which matroid** the two elements are exchangeable.

Assumption: **Independence in each matroid** can be tested.

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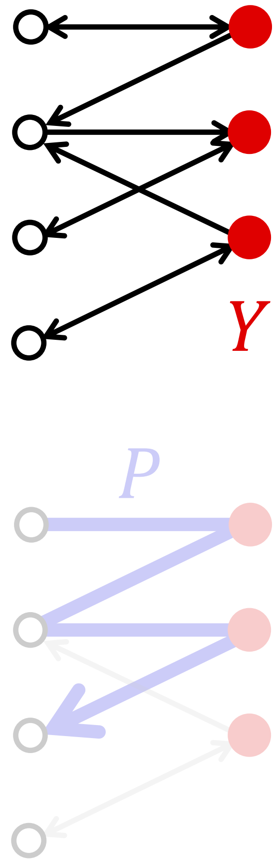

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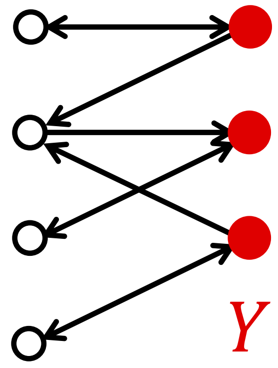

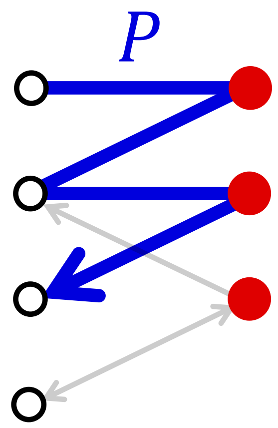

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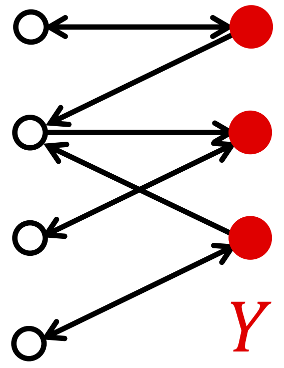

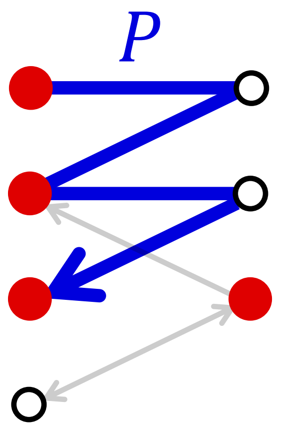

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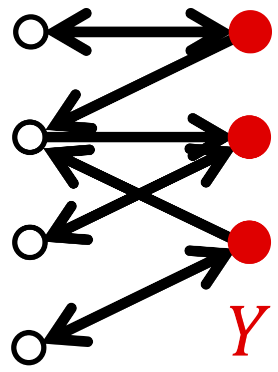

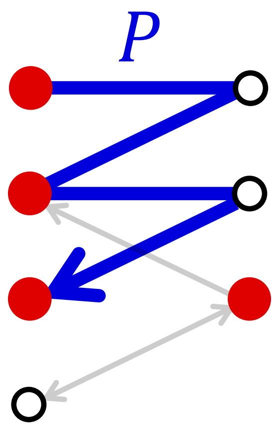

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Thm. [Edmonds 1970]

$\mathbf{M}_1, \mathbf{M}_2$: Matroids on a common ground set S

$$\max\{\, |Y| \mid Y \in \mathcal{I}_1 \cap \mathcal{I}_2 \,\} = \min\{\, r_1(Z) + r_2(S \setminus Z) \mid Z \subseteq S \,\}$$

Thm. [Frank 1981]

$\mathbf{M}_1, \mathbf{M}_2$: Matroids on a common ground set S, $w\colon S \to \mathbf{R}$

$$\max\left\{ w(Y) \,\middle|\, Y \in \mathcal{I}_1^{(k)} \cap \mathcal{I}_2^{(k)} \right\} \qquad \left(\mathcal{I}_j^{(k)} \coloneqq \{\, Y \in \mathcal{I}_j \mid |Y| = k \,\}\right)$$

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LP-relaxation (Primal)

$$\text{maximize} \sum_{e \in S} w(e)x(e)$$

$$\text{subject to} \sum_{e \in Z} x(e) \leq r_1(Z) \quad (Z \subseteq S)$$

$$\sum_{e \in Z} x(e) \leq r_2(Z) \quad (Z \subseteq S)$$

$$x(e) \geq 0 \qquad (e \in S)$$

Determine the convex hull of the common independent sets

[Edmonds 1970]

Dual LP

$$\text{minimize} \sum_{Z \subseteq S} r_1(Z)y_1(Z) + \sum_{Z \subseteq S} r_2(Z)y_2(Z)$$

$$\text{subject to} \sum_{Z \ni e} \big(y_1(Z) + y_2(Z)\big) \geq w(e) \quad (e \in S)$$

$$y_1(Z) \geq 0 \qquad (Z \subseteq S)$$

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- w is integer $\Longrightarrow$ $\exists y_i^*$: integer, optimal
- $\exists y_i^*$: optimal s.t. $\mathrm{supp}(y_i^*)$ is a chain $Z_{i,1} \subsetneq Z_{i,2} \subsetneq \cdots \subsetneq Z_{i,k}$

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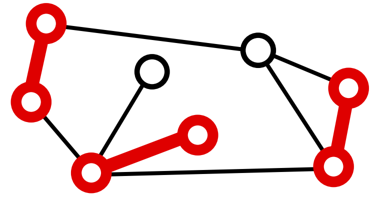
Matroid Matching generalizes Matroid Intersection and Matching

Input: $f: 2^S \to \mathbf{Z}_{\geq 0}$, 2-polymatroid function (oracle)

Goal: maximize $|Y|$ subject to $f(Y) = 2|Y|$ and $Y \subseteq S$



$G = (V, E)$, M

[Matroid Intersection] $f \coloneqq r_{\mathrm{sum}} \coloneqq r_1 + r_2$

[Matching] $f(F) \coloneqq |V(F)|$ $(F \subseteq E =: S)$

- Matroid matching is hard in general
  - Including NP-hard problems (e.g., Maximum Clique)
  - Instances for which exponentially many oracle calls are necessary [Lovász 1981; Jensen–Korte 1982]
- Tractable for linearly represented matroids [Lovász 1980, 1981; …] [Iwata–Kobayashi 2021; Pap 2013]

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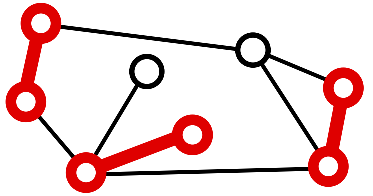

[Matroid Intersection] $f := r_{\text{sum}} := r_1 + r_2$

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What is known

- Efficient Algorithms rely on **separate** oracles for the two matroids
- Max-Min Theorems and the Polyhedral Description are given by using the two rank functions **separately**

What may be asked

- The resulting combinatorial structure is just $\mathcal{I}_1 \cap \mathcal{I}_2$
- The polytope is completely determined by $r_{\min} = \min\{r_1, r_2\}$
- When it is seen as a special case of Matroid Matching, the input should be $r_{\mathrm{sum}} = r_1 + r_2$ (oracle)

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Matroid Intersection under Restricted Oracles

Question

Is matroid intersection tractable if we only get the following information?

For each subset $X \subseteq S$,

[CI] whether $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ or not,

[MIN] $r_{\min}(X) = \min\{r_1(X), r_2(X)\}$,

[SUM] $r_{\mathrm{sum}}(X) = r_1(X) + r_2(X)$, or

[MAX] $r_{\max}(X) = \max\{r_1(X), r_2(X)\}$.

Obs. MAX is too weak as it gives no information on the second matroid if the first matroid is free, i.e., $r_1(X) = |X|$ $(\forall X \subseteq S)$.

Results and Open Problems

[CI]	whether $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ or not
[MIN]	$r_{\min}(X) = \min\{r_1(X), r_2(X)\}$
[SUM]	$r_{\text{sum}}(X) = r_1(X) + r_2(X)$
[MAX]	$r_{\max}(X) = \max\{r_1(X), r_2(X)\}$

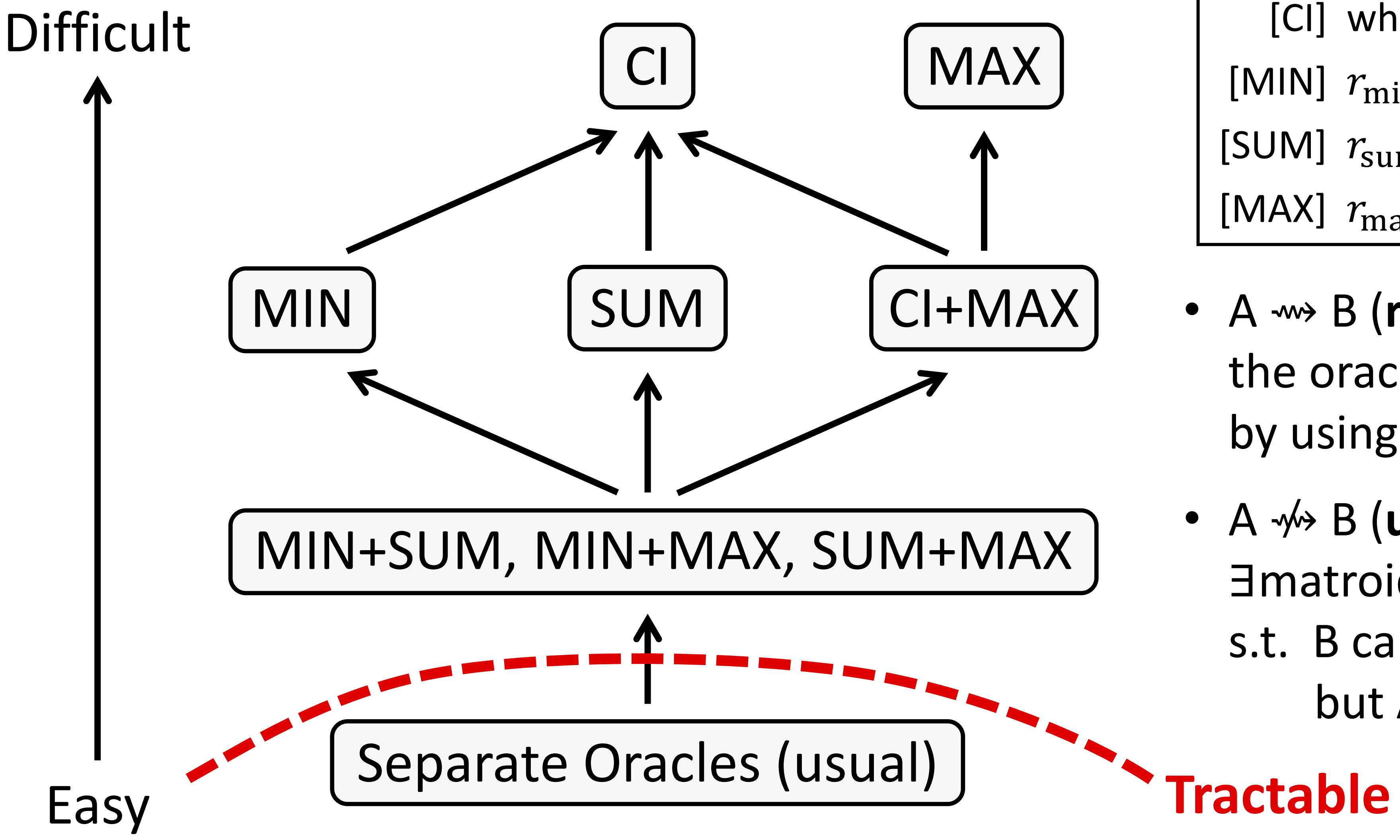
What we know (Results)

- Relation between Restricted Oracles
- SUM and CI+MAX can solve Weighted in general
- MIN can solve Unweighted in general, and Weighted in some cases
- CI can solve Unweighted/Weighted in some cases

What we want to know (Open)

- Can MIN solve Weighted in general? Or, is it hard?
- Can CI solve Unweighted/Weighted in general? Or, is it hard?

[Result 1] Relation between Restricted Oracles



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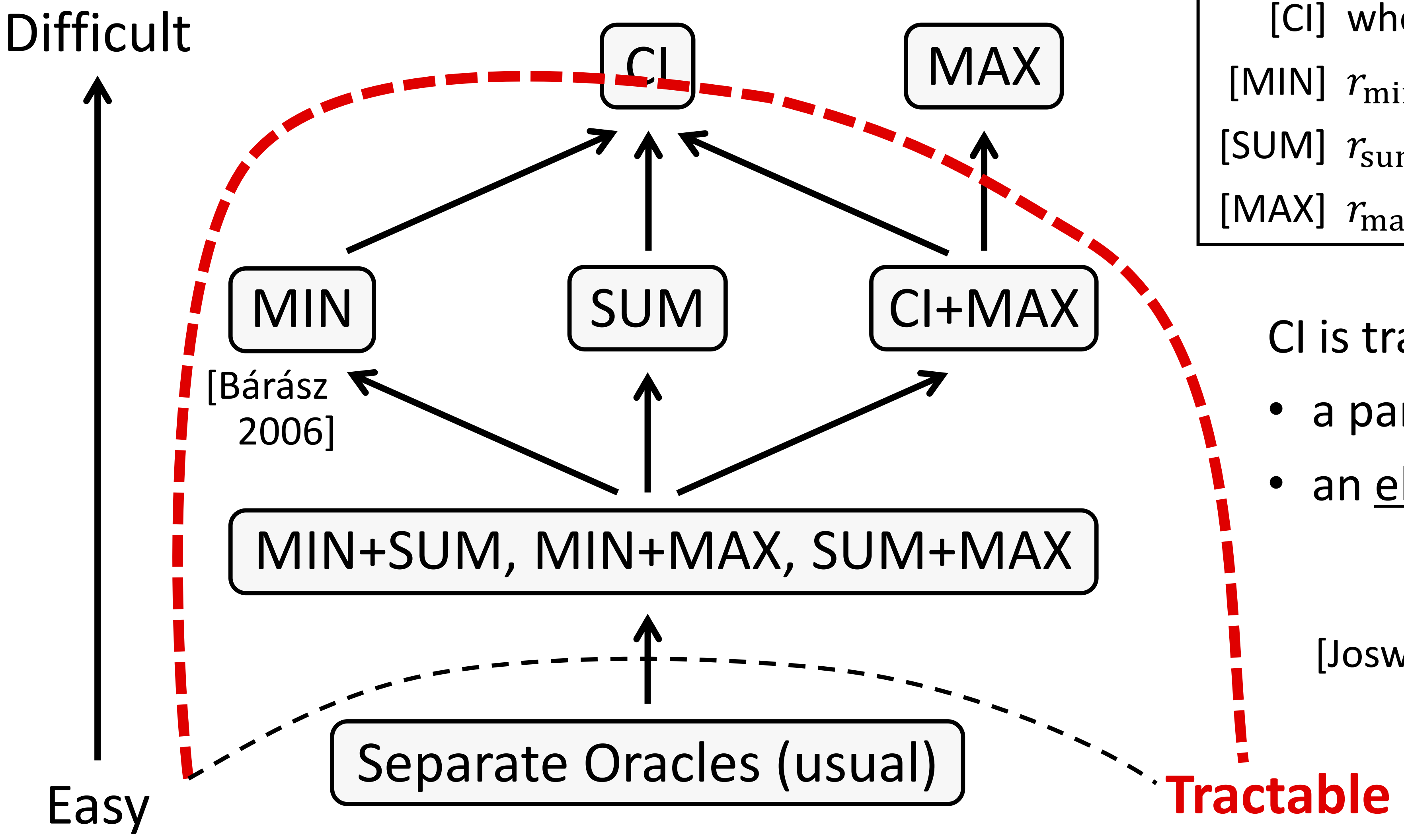
- A $\rightsquigarrow$ B (**reachable**) means the oracle B is always emulated by using the oracle A
- A $\not\rightsquigarrow$ B (**unreachable**) means ∃matroid intersection instances s.t. B can distinguish them but A cannot

[Result 2] Unweighted Matroid Intersection



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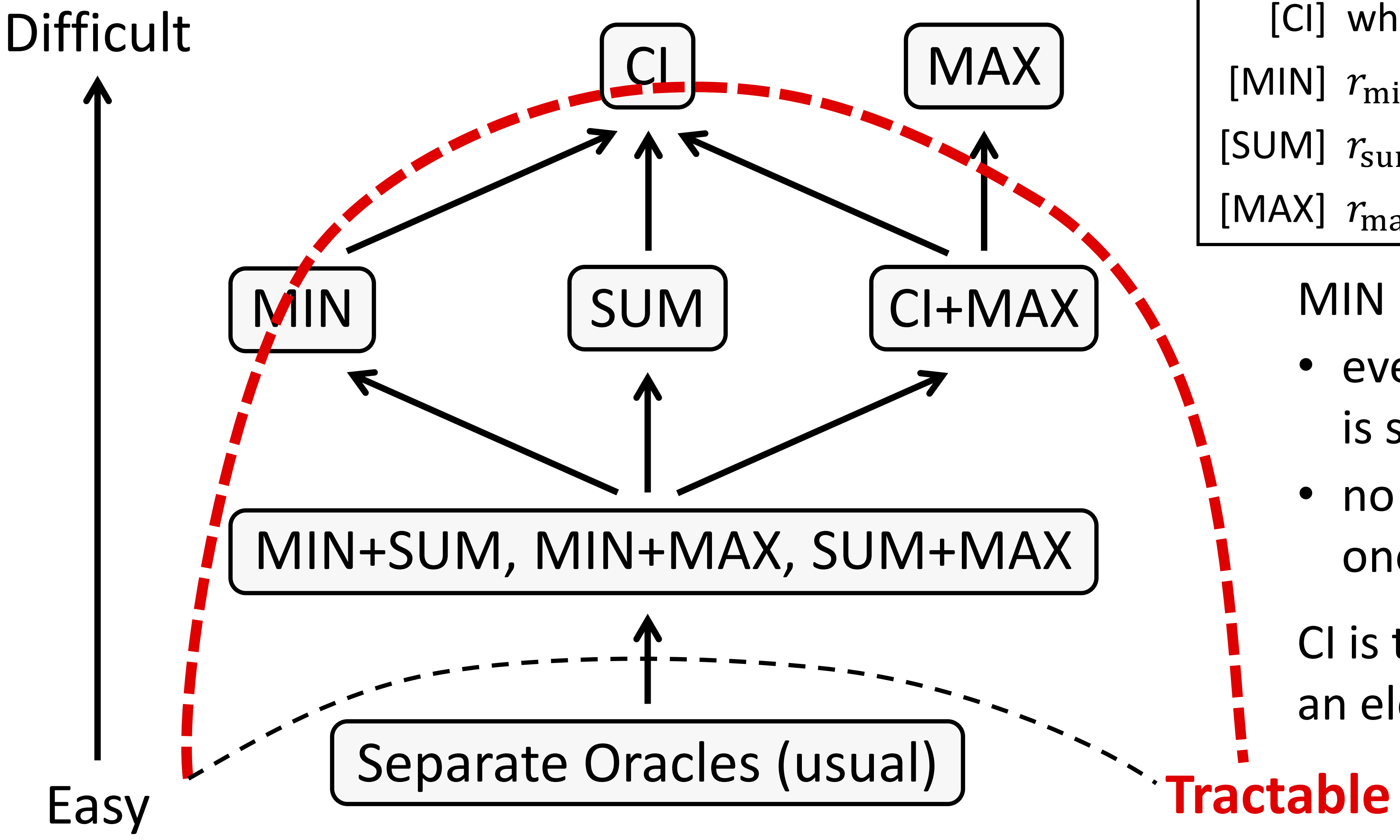
CI is tractable if one matroid is

- a partition matroid, or
- an <u>elementary split matroid</u>.

Generalization of paving matroid

[Joswig–Schröter 2017; BKSYY 2023]

[Result 3] Weighted Matroid Intersection



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[MAX] $r_{\max}(X) = \max\{r_1(X), r_2(X)\}$

MIN is tractable if

- every circuit in one matroid is small (FPT), or
- no pair of circuits s.t. one includes the other.

CI is tractable if one matroid is an elementary split matroid.

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Matroid (Notation)

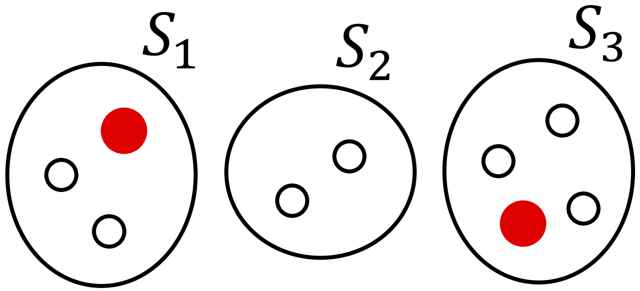
$\mathbf{M}$: Matroid on a ground set S

- $\mathcal{I} \subseteq 2^S$: **Independent set** family
- $\mathcal{B} \subseteq 2^S$: **Base (Basis)** family
- $\mathcal{C} \subseteq 2^S$: **Circuit** family
- $r: 2^S \to \mathbf{Z}_{\geq 0}$, **Rank** function; $r(X) \coloneqq \max\{\, |Y| \mid Y \subseteq X,\ Y \in \mathcal{I}\,\}$
- $\mathrm{cl}: 2^S \to 2^S$, **Closure** operator; $\mathrm{cl}(X) \coloneqq \{\, e \in S \mid r(X \cup \{e\}) = r(X)\,\}$

Matroid (Examples)

- Partition Matroid

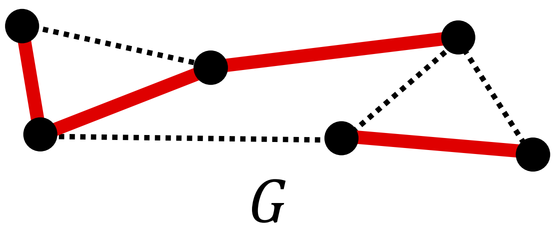
  $S = S_1 \uplus S_2 \uplus \cdots \uplus S_k,\ \mathcal{I} = \{\, Y \subseteq S \mid |Y \cap S_i| \leq 1\ (\forall i \in [k]) \,\}$



- Cycle Matroid (Graphic Matroid)

  $G = (V, E)$: undirected graph

  $S = E,\ \mathcal{I} = \{\, Y \subseteq S \mid Y \text{ forms a forest (contains no cycle)} \,\}$



- Paving Matroid
  - Nearly Uniform: $\forall C \in \mathcal{C},\ r(S) \leq |C| \leq r(S) + 1$
  - Hypergraph Representation: $\exists r \in \mathbf{Z}_{>0},\ \exists \mathcal{H} \subseteq \binom{S}{\geq r}$,

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    $\mathcal{B} = \{\, Y \subseteq S \mid |Y| = r,\ Y \nsubseteq H\ (\forall H \in \mathcal{H}) \,\}$

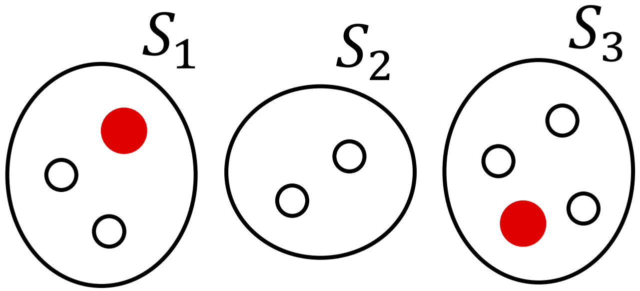
↓Generalized

Elementary Split Matroid [BKSYY 2023]

Matroid (Examples)

- Partition Matroid

  $S = S_1 \uplus S_2 \uplus \cdots \uplus S_k,\ \mathcal{I} = \{\, Y \subseteq S \mid |Y \cap S_i| \leq 1\ (\forall i \in [k]) \,\}$



- Cycle Matroid (Graphic Matroid)

  $G = (V, E)$: undirected graph

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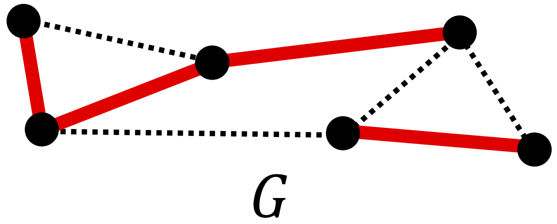

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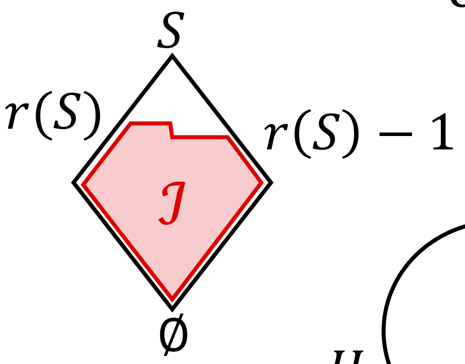

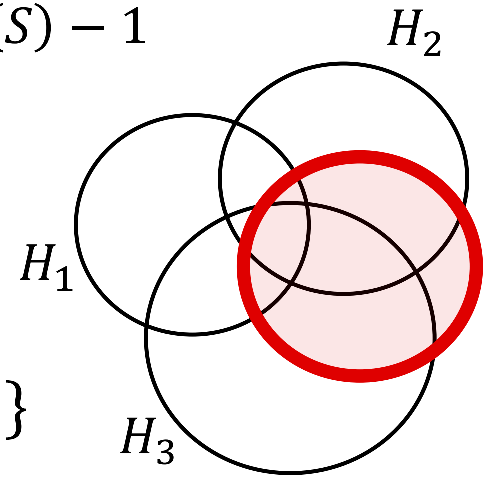

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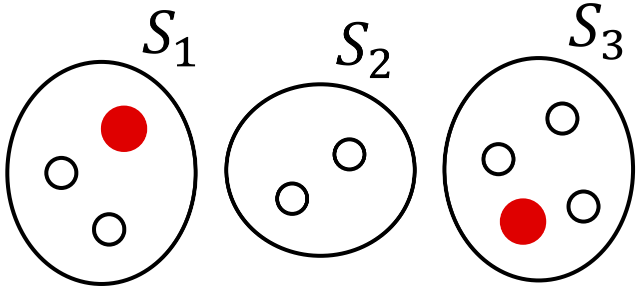

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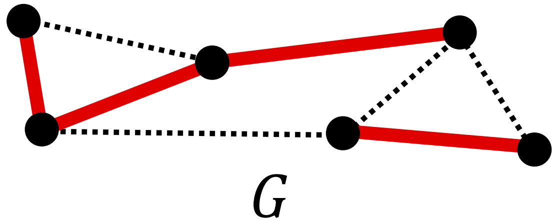

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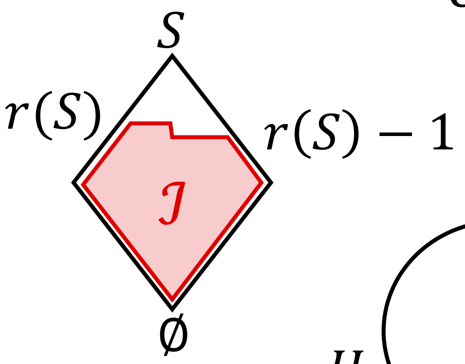
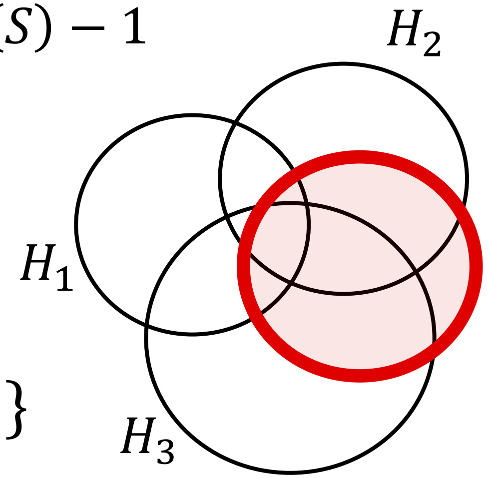
Elementary Split Matroid

[BKSYY 2023]





Matroid Intersection Problem (Unweighted)

Input: S: Finite set, $\mathbf{M}_1, \mathbf{M}_2$: Matroids on S (**oracle**)
Goal: maximize $\lvert Y \rvert$ subject to $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$

- Usually, **separate oracles** are given, i.e., we can ask for each subset $X \subseteq S$ and each $i = 1, 2$, whether $X \in \mathcal{I}_i$ or not, the rank $r_i(X)$, etc.
- Many Applications (Special Cases)
  - Bipartite matching: Partition + Partition
  - Arborescence (packing): Partition + Graphic (unions)
  - Dijoin: Partition + Crossing Submodular Function [Frank–Tardos 1981]

Matroid Intersection Problem (Weighted)

Input: S: Finite set, $\mathbf{M}_1, \mathbf{M}_2$: Matroids on S (**oracle**), $w: S \to \mathbf{R}$

Goal: maximize $w(Y)$ subject to $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ (and $|Y| = k$ for each k)

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  we can ask for each subset $X \subseteq S$ and each $i = 1, 2$,
  whether $X \in \mathcal{J}_i$ or not, the rank $r_i(X)$, etc.

- The goal of this study is to clarify what happens if **the oracle is restricted**:

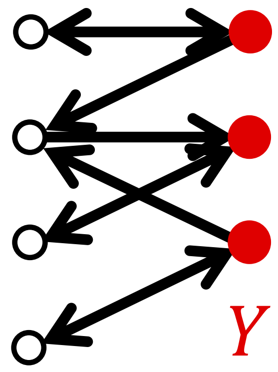
[CI] whether $X \in \mathcal{J}_1 \cap \mathcal{J}_2$ or not,
[MIN] $r_{\min}(X) = \min\{r_1(X), r_2(X)\}$, or
[SUM] $r_{\text{sum}}(X) = r_1(X) + r_2(X)$.

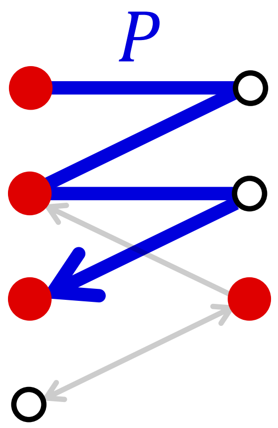
“Matroid Intersection is **Tractable**”

Most of them require separate information of two matroids

Basic Strategy of Efficient Algorithms

- Starting with $Y = \emptyset$, repeatedly update the current solution Y.
- For each update,
  - construct the exchangeability graph w.r.t. Y,
  - find an augmenting path P in the graph, and
  - flip the current solution along the path, i.e., $Y \leftarrow Y \triangle P$.
- The edges in the graph are oriented according to **in which matroid** the two elements are exchangeable.





Assumption: **Independence in each matroid** can be tested.

Exchangeability Graph

Def. $\mathbf{M}_1, \mathbf{M}_2$: Matroids on a common ground set S, $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$

$D[Y] = (S \setminus Y, Y; A[Y])$: **Exchangeability Graph** w.r.t. Y

$$\Updownarrow^{\text{def}}$$

- $A[Y] = A_1[Y] \cup A_2[Y]$, where
  - $A_1[Y] := \{ (y, x) \mid Y - y + x \in \mathcal{I}_1 \}$
  - $A_2[Y] := \{ (x, y) \mid Y - y + x \in \mathcal{I}_2 \}$
- $S_1 := \{ x \mid Y + x \in \mathcal{I}_1 \}$ (Sources)
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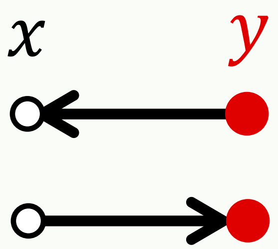
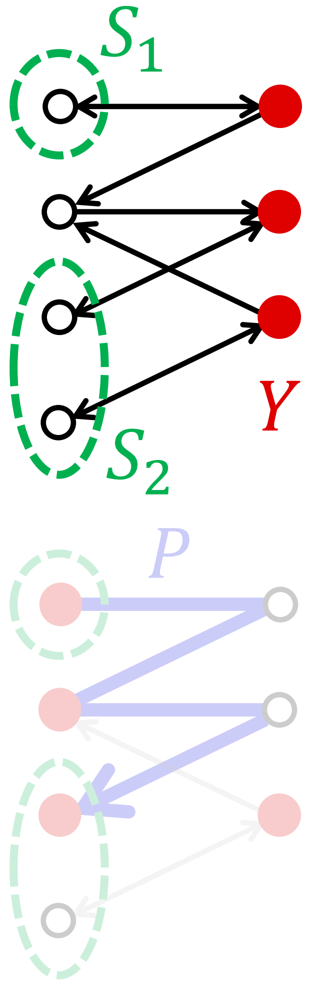
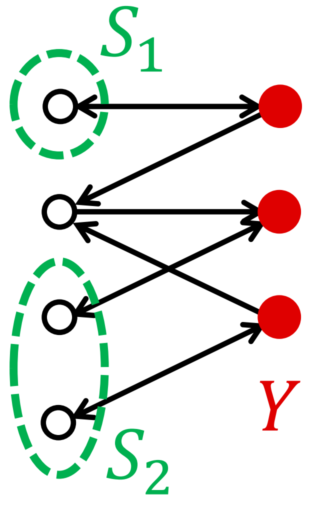




Augmentability (Unweighted)

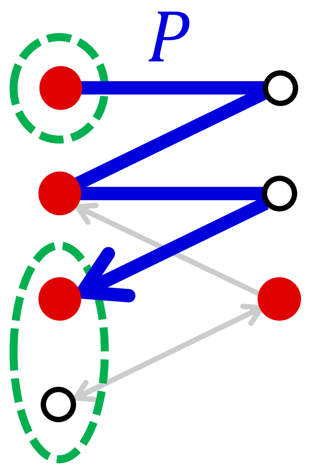
Thm. $\mathbf{M}_1, \mathbf{M}_2$: Matroids on a common ground set S, $Y \in \mathcal{J}_1 \cap \mathcal{J}_2$
$D[Y] = (S \setminus Y, Y; A[Y])$: Exchangeability Graph w.r.t. Y

- If $D[Y]$ has no S_1–S_2 path, then $|Y|$ is maximum.
- If P is a **shortest** S_1–S_2 path in $D[Y]$, then $Y \triangle P \in \mathcal{J}_1 \cap \mathcal{J}_2$.



$\mathrm{O}(nr^2)$ time in total, where $r :=$ opt. value $\leq n$

- $D[Y]$ is constructed by $\mathrm{O}(nr)$ oracle calls
- P is found by BFS in linear time (n vertices, $\mathrm{O}(nr)$ edges)
- #(iteration) is $r + 1$



Augmentability (Weighted)

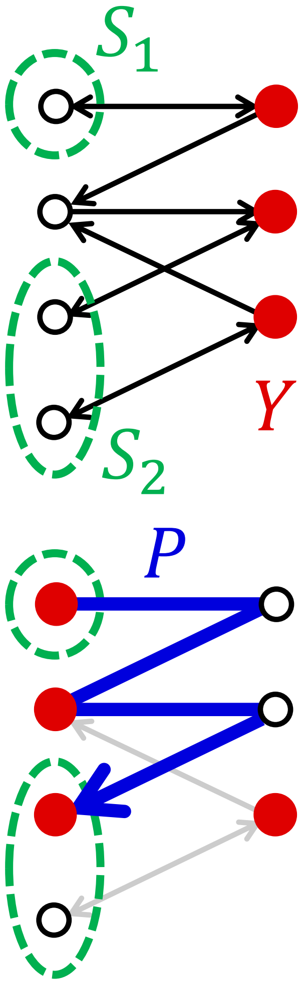
Thm. $\mathbf{M}_1, \mathbf{M}_2$: Matroids on a common ground set S

$$Y \in \operatorname{argmax}\left\{ w(X) \;\middle|\; X \in \mathcal{I}_1^{(k)} \cap \mathcal{I}_2^{(k)} \right\} \quad (k = |Y|)$$

$D[Y] = (S \setminus Y, Y; A[Y])$: Exchangeability Graph w.r.t. Y

$$\mathrm{cost}(P) \coloneqq w(P \cap Y) - w(P \setminus Y) \quad (P\text{: path/cycle})$$

- $D[Y]$ has **no negative-cost cycle**.
- If P is a **shortest cheapest** S_1–S_2 path in $D[Y]$, then $Y \triangle P \in \operatorname{argmax}\left\{ w(X) \;\middle|\; X \in \mathcal{I}_1^{(k+1)} \cap \mathcal{I}_2^{(k+1)} \right\}$.



$O(n^2 r^2)$ time in total (Bellman–Ford, Weight Splitting, etc.)

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Exchangeability Graph via Restricted Oracles

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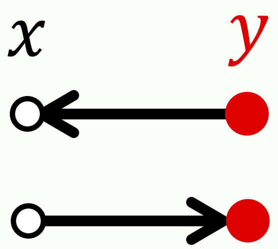

- $S_1 \coloneqq \{ x \mid Y + x \in \mathcal{J}_1 \}$ (Sources)
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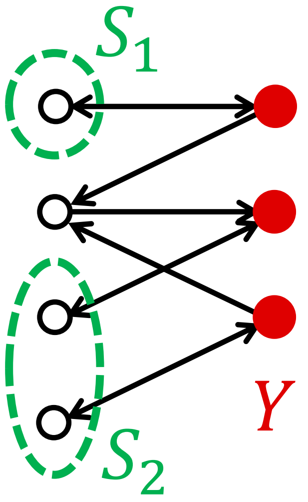

Sources and Sinks via Restricted Oracles

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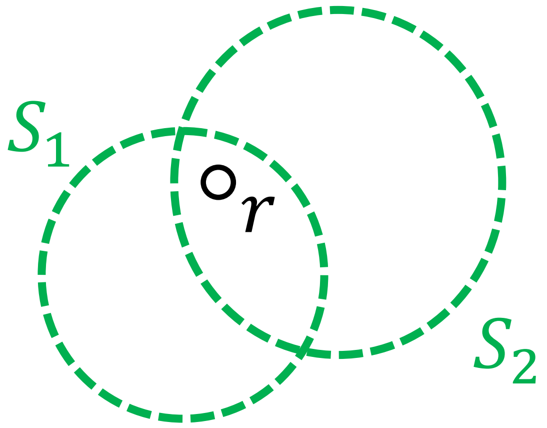
- $r \in S_1 \cap S_2 \iff Y + r \in \mathcal{I}_1 \cap \mathcal{I}_2$

  This can be recognized by CI, and hence by MIN or SUM.



- $s \in S_1 \setminus S_2,\ t \in S_2 \setminus S_1 \iff$
  - $r_1(Y + s) = |Y| + 1,\ r_2(Y + s) = |Y|$
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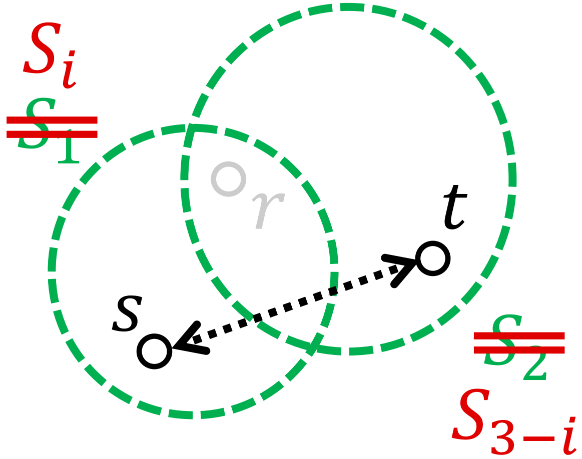

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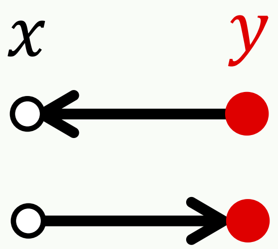

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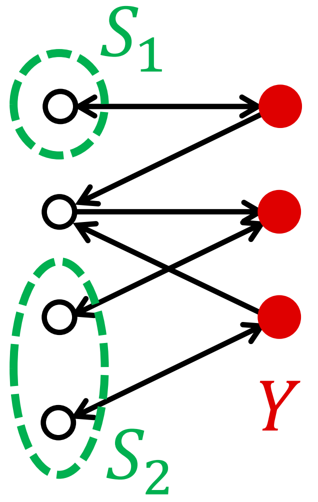

Exchange Edges via Restricted Oracles

$A_1[Y] \coloneqq \{\, (y, x) \mid Y - y + x \in \mathcal{I}_1 \,\}$

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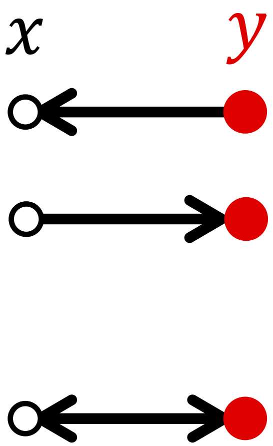

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  - SUM (or CI+MAX) is strong enough to emulate Bellman–Ford (Weighted)
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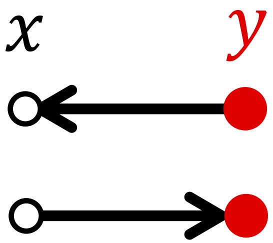

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Unweighted Matroid Intersection via MIN [Bárász 2006]

Emulate a usual algorithm on the **Overestimated** Exchangeability Graph

- Sources, Sinks, and Edges around them are correctly recognized (up to sym.).
- Other edges are overestimated so that if an edge e is wrongly estimated, there is a correct **shortcut** skipping e.

→ **None of such fake edges is used in a shortest path!**

Thm. $Y \in \mathcal{I}_1 \cap \mathcal{I}_2,\ D[Y]$: Exchangeability Graph w.r.t. Y

- If $D[Y]$ has no S_1–S_2 path, then $|Y|$ is maximum.
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Overestimation of $A[Y]$ via MIN (Unweighted)

$S_i = \{ x \mid Y + x \in \mathcal{I}_i \}$ $(i = 1, 2)$
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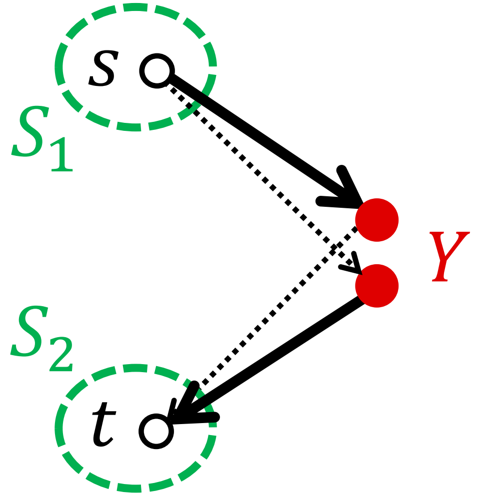
Assume $S_1 \cap S_2 = \emptyset,\ s \in S_1,\ t \in S_2$

- $Y + s \in \mathcal{I}_1 \setminus \mathcal{I}_2,\ Y + t \in \mathcal{I}_2 \setminus \mathcal{I}_1$
- $\forall y \in Y,$
  - $(y, t) \in A_1[Y] \iff r_1(Y - y + t) = |Y| \iff r_{\min}(Y - y + t) = |Y|$
  - $(s, y) \in A_2[Y] \iff r_2(Y - y + t) = |Y| \iff r_{\min}(Y - y + s) = |Y|$
- Suppose that $\forall x \in S \setminus (Y \cup S_1 \cup S_2), \forall y \in Y,$
  - Estimate $\exists (y, x) \iff (y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$
  - Estimate $\exists (x, y) \iff (x, y) \in A_2[Y]$ or $(s, y) \in A_2[Y]$

Enough to find a shortest path! (Wrong $\Longrightarrow$ ∃Shortcut)



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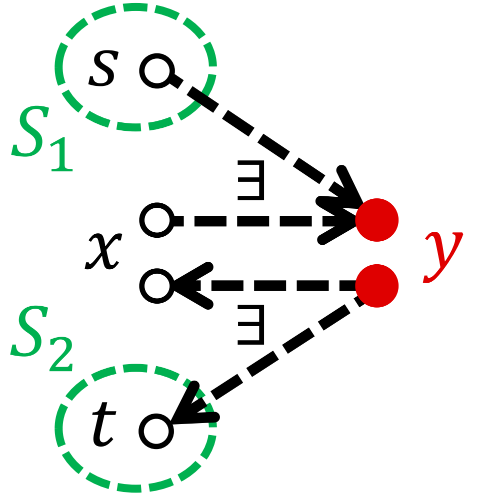
  - $(y, t) \in A_1[Y] \iff r_1(Y - y + t) = |Y| \iff r_{\min}(Y - y + t) = |Y|$
  - $(s, y) \in A_2[Y] \iff r_2(Y - y + t) = |Y| \iff r_{\min}(Y - y + s) = |Y|$
- Suppose that $\forall x \in S \setminus (Y \cup S_1 \cup S_2), \forall y \in Y,$
  - Estimate $\exists (y, x) \iff (y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$
  - Estimate $\exists (x, y) \iff (x, y) \in A_2[Y]$ or $(s, y) \in A_2[Y]$

Enough to find a shortest path! (Wrong $\implies$ $\exists$Shortcut)



Overestimation of $A[Y]$ via MIN (Unweighted)

$S_i = \{ x \mid Y + x \in \mathcal{I}_i \}$ $(i = 1, 2)$
$A_1[Y] = \{ (y, x) \mid Y - y + x \in \mathcal{I}_1 \}$
$A_2[Y] = \{ (x, y) \mid Y - y + x \in \mathcal{I}_2 \}$

Assume $S_1 \cap S_2 = \emptyset,\ s \in S_1,\ t \in S_2$

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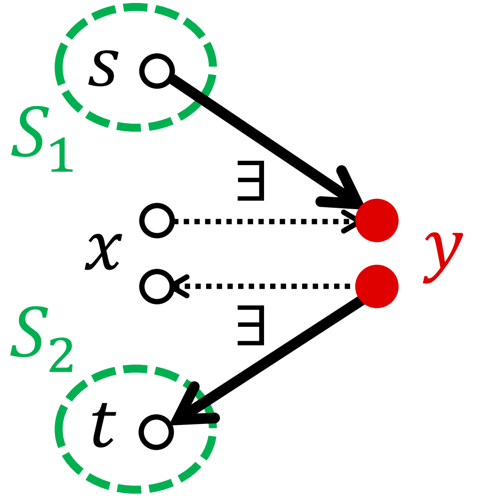

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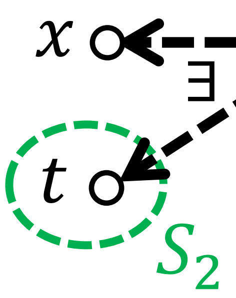
Assume $S_1 \cap S_2 = \emptyset,\ s \in S_1,\ t \in S_2$

- $Y + s \in \mathcal{I}_1 \setminus \mathcal{I}_2,\ Y + t \in \mathcal{I}_2 \setminus \mathcal{I}_1$
- Estimate $\exists (y, x)\ \ (x \in S \setminus (Y \cup S_1 \cup S_2), y \in Y)$

$\Leftrightarrow\ (y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$

$\Leftrightarrow\ r_1(Y - y + x) = |Y|$ or $r_1(Y - y + t) = |Y|$



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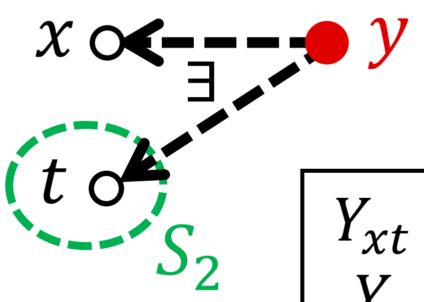

$\Leftrightarrow\ (y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$

$\Leftrightarrow\ r_1(Y - y + x) = |Y|$ or $r_1(Y - y + t) = |Y|$

$\Leftrightarrow\ r_1(Y - y + x + t) = |Y|$

$Y_{xt} := Y + x + t$
$Y_x := Y + x$
$Y_t := Y + t$

$$\left(\begin{array}{l} \underbrace{|Y| = r_1(Y_{xt})}_{x, t \in S \setminus S_1 = \mathrm{cl}_1(Y)} \geq r_1(Y_{xt} - y) \geq \max\{r_1(Y_x - y), r_1(Y_t - y)\} \\ r_1(Y_x - y) + r_1(Y_t - y) \geq r_1(Y_{xt} - y) + \underbrace{r_1(Y - y)}_{= |Y| - 1} \end{array} \right)$$

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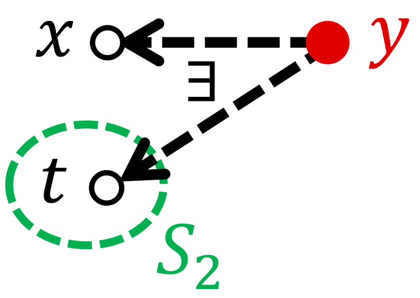
$\Leftrightarrow\ (y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$

$\Leftrightarrow\ r_1(Y - y + x) = |Y|$ or $r_1(Y - y + t) = |Y|$

$\Leftrightarrow\ r_1(Y - y + x + t) = |Y|$

$\Leftrightarrow\ r_{\min}(Y - y + x + t) = |Y|$

$(r_2(Y - y + x + t) \geq r_2(Y + t) - 1 = |Y|)$



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Assume $S_1 \cap S_2 = \emptyset$, $s \in S_1$, $t \in S_2$

- $Y + s \in \mathcal{I}_1 \setminus \mathcal{I}_2$, $Y + t \in \mathcal{I}_2 \setminus \mathcal{I}_1$
- Estimate $\exists (y, x)$ $(x \in S \setminus (Y \cup S_1 \cup S_2), y \in Y)$
  $\Leftrightarrow$ $(y, x) \in A_1[Y]$ or $(y, t) \in A_1[Y]$
  $\Leftrightarrow$ $r_1(Y - y + x) = |Y|$ or $r_1(Y - y + t) = |Y|$
  $\Leftrightarrow$ $r_1(Y - y + x + t) = |Y|$
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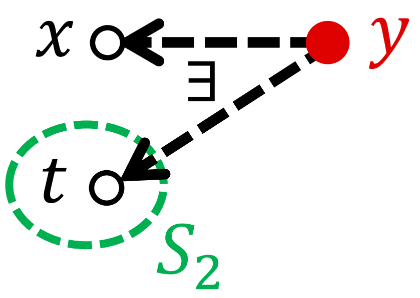

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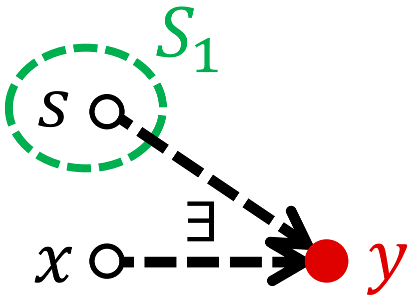

Unweighted Matroid Intersection via MIN [Bárász 2006]

Emulate a usual algorithm on the **Overestimated** Exchangeability Graph

- Sources, Sinks, and Edges around them are correctly recognized (up to sym.).
- Other edges are overestimated so that
  if an edge e is wrongly estimated, there is a correct **shortcut** skipping e.

  → **None of such fake edges is used in a shortest path!**

Thm. $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$, $D[Y]$: Exchangeability Graph w.r.t. Y

- If $D[Y]$ has no S_1–S_2 path, then $|Y|$ is maximum.
- If P is a **shortest** S_1–S_2 path in $D[Y]$, then $Y \triangle P \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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  $\rightarrow$ **None of such fake edges is used in a shortest path!**
- When the algorithm halts, a dual optimal solution is found by reachability.

Thm. $\max\{\, |Y| \mid Y \in \mathcal{I}_1 \cap \mathcal{I}_2 \,\} = \min\{\, r_{\min}(Z) + r_{\min}(S \setminus Z) \mid Z \subseteq S \,\}$

$$|Y| = \underbrace{r_{\min}(Y \cap Z)}_{= |Y \cap Z|} + \underbrace{r_{\min}(Y \setminus Z)}_{= |Y \setminus Z|} \leq r_{\min}(Z) + r_{\min}(S \setminus Z) \leq r_1(Z) + r_2(S \setminus Z)$$

Weighted Matroid Intersection via MIN [BBKYY 2023+]

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- Sources, Sinks, and Edges around them are correctly recognized (up to sym.).
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  - Such **fake edges may be used** in a (shortest) cheapest path!
  - They **may cause negative-cost cycles**! (NP-hard!?)

Thm. $Y \in \mathcal{I}_1^{(k)} \cap \mathcal{I}_2^{(k)}$: Max-Weight, $\mathrm{cost}(P) \coloneqq w(P \cap Y) - w(P \setminus Y)$ (P: path/cycle)

- $D[Y]$ has **no negative-cost cycle**.
- If P is a **shortest cheapest** S_1–S_2 path in $D[Y]$, then $Y \triangle P \in \mathcal{I}_1^{(k+1)} \cap \mathcal{I}_2^{(k+1)}$ is max-weight.

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- **Any graph consistent** with all the extra information **is suitable** for emulation!!
- Finding a consistent graph is **NP-hard**… (4-coloring of 3-colorable graphs)

Extra Info. from Two-by-Two Exchange [BBKYY 2023+]

Extra information from **at most two-by-two exchanges** may refine the graph!

- $r_{\min}(Y - y + x) = |Y| \iff$

- Otherwise,
  - $r_{\min}(Y - y_1 - y_2 + x) = |Y| - 1 \iff$
  - $r_{\min}(Y - y + x_1 + x_2) = |Y| \iff$

- Otherwise (none of the above holds),

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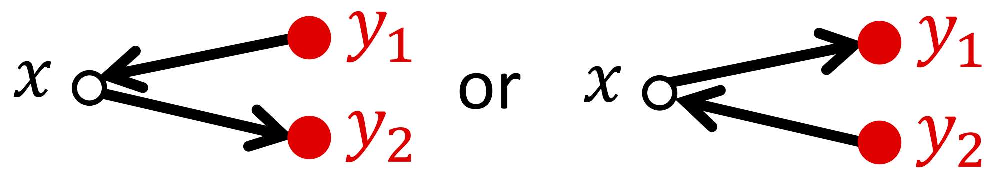

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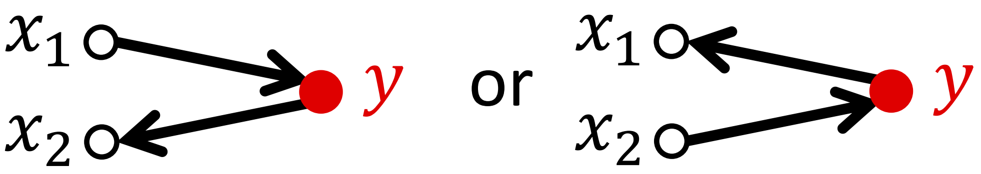

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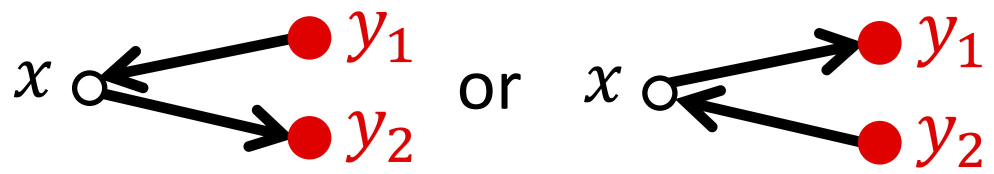
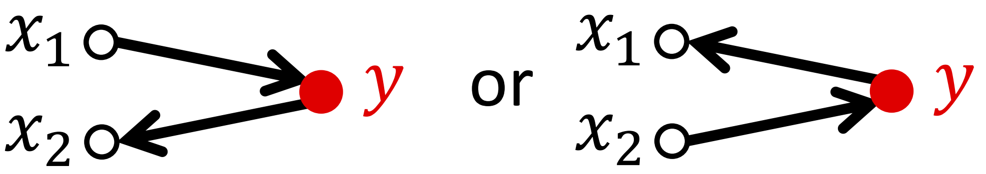
  - $r_{\min}(Y - y_1 - y_2 + x) = |Y| - 1 \iff$ 

  - $r_{\min}(Y - y + x_1 + x_2) = |Y| \iff$ 

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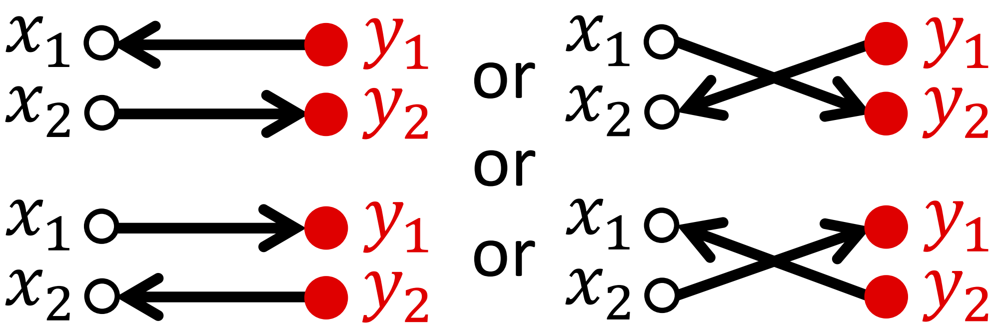
$r_{\min}(Y - y_1 - y_2 + x_1 + x_2) = |Y| - 1 \iff$ 

Weighted Matroid Intersection via MIN [BBKYY 2023+]

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- **Extra information** from **at most two-by-two exchanges** may refine the graph!
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- Finding a consistent graph is **NP-hard**... (4-coloring of 3-colorable graphs)

Consistency with Extra Info. is Enough [BBKYY 2023+]

Any graph consistent with all the extra information **is suitable** for emulation!!

Thm. $Y \in \operatorname{argmax}\left\{ w(X) \,\middle|\, X \in \mathcal{I}_1^{(k)} \cap \mathcal{I}_2^{(k)} \right\}$ $(k = |Y|)$

$D[Y]$: Exchangeability Graph w.r.t. Y

$\tilde{D}[Y]$: Subgraph of the overestimation **consistent with all the extra info**.

$\mathrm{cost}(P) \coloneqq w(P \cap Y) - w(P \setminus Y)$ $(P\text{: path/cycle})$

- $\tilde{D}[Y]$ has **no negative-cost cycle**.
- $\forall \tilde{P}$: **shortest cheapest** S_1–S_2 path in $\tilde{D}[Y]$,
  $\exists P$: **shortest cheapest** S_1–S_2 path in $D[Y]$ with the same vertex set,
  and vice versa.

Weighted Matroid Intersection via MIN [BBKYY 2023+]

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Tractable cases of WMI via MIN [BBKYY 2023+]

- When $\forall C_1 \in \mathcal{C}_1, \forall C_2 \in \mathcal{C}_2,\ C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$
  $\rightarrow$ Finding a consistent graph is reduced to **2-SAT**
- When $\exists i \in \{1,2\}, \forall C \in \mathcal{C}_i,\ |C| \leq k$
  $\rightarrow$ $O\left(2^k \cdot \mathrm{poly}(n)\right)$ time by 2-SAT + Brute-Force Guess
- **Lexicographical Maximization**
  - Max. #(heaviest); Sub. to this, Max. #(second heaviest); and so on
  - Update **with preserving the numbers of heavier elements** can be done via **Underestimation** of the Exchangeability Graph (by 2-SAT)
  - **Approximation with factor 2 or better** for the original problem [BKYY 2022]

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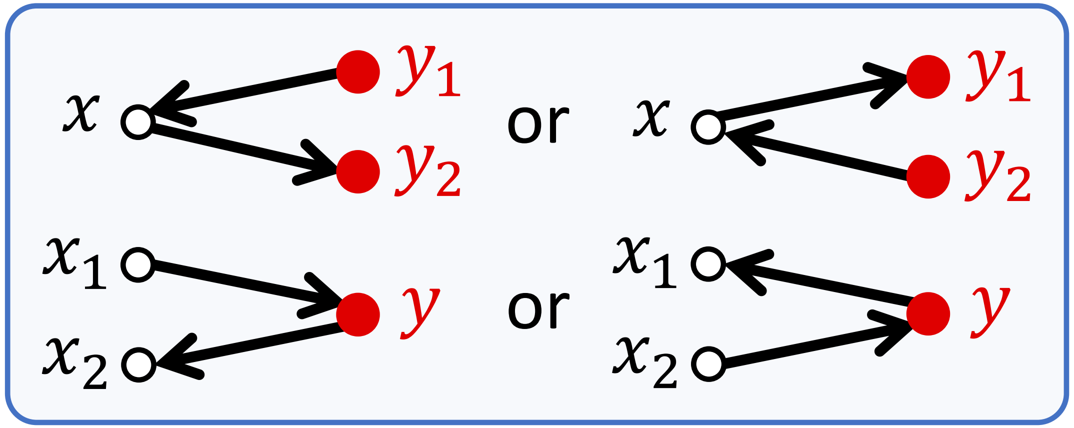
2-SAT works! (Easy)



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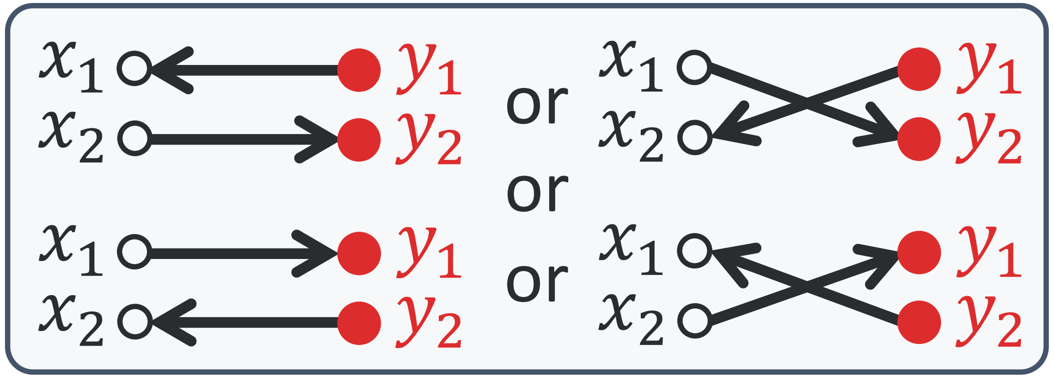

Can represent 4-coloring (Hard)

Tractable cases of WMI via MIN [BBKYY 2023+]

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Summary

Question

Is matroid intersection tractable if we only get the following information?

For each subset $X \subseteq S$,

[CI] whether $X \in \mathcal{J}_1 \cap \mathcal{J}_2$ or not,

[MIN] $r_{\min}(X) = \min\{r_1(X), r_2(X)\}$,

[SUM] $r_{\text{sum}}(X) = r_1(X) + r_2(X)$, or

[MAX] $r_{\max}(X) = \max\{r_1(X), r_2(X)\}$.

Obs. MAX is too weak as it gives no information on the second matroid if the first matroid is free, i.e., $r_1(X) = |X|$ $(\forall X \subseteq S)$.

Summary

[CI]	whether $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ or not
[MIN]	$r_{\min}(X) = \min\{r_1(X), r_2(X)\}$
[SUM]	$r_{\mathrm{sum}}(X) = r_1(X) + r_2(X)$
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<u>What we know (Results)</u>

- Relation between Restricted Oracles
- SUM and CI+MAX can solve Weighted in general (Emulate Bellman–Ford)
- MIN can solve Unweighted in general, and Weighted in some cases
  - No circuit inclusion (2-SAT)
  - All circuits are small in one matroid (2-SAT + Brute-Force Guess, FPT)
  - Lexicographical Maximization (2-SAT, 2- or better Approximation in general)
- CI can solve Unweighted/Weighted in some cases
  - One is a partition matroid, Unweighted (Emulate BFS)
  - One is an elementary split matroid, Weighted (Brute-Force)

Summary

[CI]	whether $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ or not
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  - Lexicographical Maximization (2-SAT, 2- or better Approximation in general)
- CI can solve Unweighted/Weighted in some cases
  - One is **partition or chain (general upper bounds)**, Unweighted (Emulate BFS)
  - One is an elementary split matroid, Weighted (Brute-Force)

Summary

[CI]	whether $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ or not
[MIN]	$r_{\min}(X) = \min\{r_1(X), r_2(X)\}$
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What we know (Results)

- Relation between Restricted Oracles
- SUM and CI+MAX can solve Weighted in general
- MIN can solve Unweighted in general, and Weighted in some cases
- CI can solve Unweighted/Weighted in some cases

What we want to know (Open)

- Can MIN solve Weighted in general? Or, is it hard?
- Can CI solve Unweighted/Weighted in general? Or, is it hard?